

Quantum Isometries of the finite noncommutative geometry of the Standard Model

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Abstract

We compute the quantum isometry group of the finite noncommutative geometry F describing the internal degrees of freedom in the Standard Model of particle physics. We show that this provides genuine quantum symmetries of the spectral triple corresponding to $M \times F$, where M is a compact spin manifold. We also prove that the bosonic and fermionic part of the spectral action are preserved by these symmetries.

1 Introduction

In modern theoretical physics, symmetries play a fundamental role in determining the dynamics of a theory. In the two foremost examples, namely General Relativity and the Standard Model of elementary particles, the dynamics is dictated by invariance under diffeomorphisms and under local gauge transformations respectively. As a way to unify external (i.e. diffeomorphisms) and internal (i.e. local gauge) symmetries, Connes and Chamseddine proposed a model from Noncommutative Geometry [15] based on the product of the canonical commutative spectral triple of a compact Riemannian spin manifold M and a finite dimensional noncommutative one, describing an “internal” finite noncommutative space F [12, 13, 18, 20]. In this picture, diffeomorphisms are realized as outer automorphisms of the algebra, while inner automorphisms correspond to the gauge transformations. Inner fluctuations of the Dirac operator are divided in two classes: the 1-forms coming from commutators with the Dirac operator of M give the gauge bosons, while the 1-forms coming from the Dirac operator of F give the Higgs field. The gravitational and bosonic part S_b of the action is encoded in the spectrum of the gauged Dirac operator, which is invariant under isometries of the Hilbert space. The fermionic part S_f is also defined in terms of the spectral data. The result is an Euclidean version of the Standard Model minimally coupled to gravity (cf. [20] and references therein).

In his “Erlangen program”, Klein linked the study of geometry with the analysis of its group of symmetries. Dealing with quantum geometries, it is natural to study quantum symmetries. The idea of using quantum group symmetries to understand the conceptual significance of the finite geometry F is mentioned in a final remark by Connes in [17]. Preliminary studies on

the Hopf-algebra level appeared in [30, 21, 26]. Following Connes' suggestion, quantum automorphisms of finite-dimensional complex C^* -algebras were introduced by Wang in [37, 38] and later the quantum permutation groups of finite sets and graphs have been studied by a number of mathematicians, see e.g. [3, 4, 11, 34]. These are compact quantum groups in the sense of Woronowicz [41]. The notion of compact quantum symmetries for “continuous” mathematical structures, like commutative and noncommutative manifolds (spectral triples), first appeared in [28], where quantum isometry groups were defined in terms of a Laplacian, followed by the definition of “quantum groups of orientation preserving isometries” based on the theory of spectral triples in [7], and on spectral triples with a real structure in [29]. Computations of these compact quantum groups were done for several examples, including the tori, spheres, Podleś quantum spheres, and Rieffel deformations of compact Riemannian spin manifolds. For these studies we refer to [6, 7, 8, 9, 10] and references therein.

The finite noncommutative geometry $F = (A_F, H_F, D_F, \gamma_F, J_F)$ describing the internal space of the Standard Model is given by a unital real spectral triple over the finite-dimensional real C^* -algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, with \mathbb{H} the field of quaternions. Let $B_F \subset \mathcal{B}(\mathcal{H})$ be the smallest complex C^* -algebra containing A_F as a real C^* -subalgebra. In this article we first compute the quantum group of orientation and real structure preserving isometries of the spectral triple $(B_F, H_F, D_F, \gamma_F, J_F)$; next we show that this quantum symmetry can be extended to get quantum isometries of the product of this spectral triple with the canonical spectral triple of M . Thus, we have genuine quantum symmetries of the full spectral triple of the Standard Model. Moreover these quantum symmetries preserves the spectral action in a suitable sense. Finally we compute the maximal quantum subgroup of the quantum isometry group whose coaction is a quantum automorphism of the real C^* -algebra A_F .

The plan of this article is as follows. We start by recalling in Sec. 2 some basic definitions and facts about compact quantum groups and quantum isometries. In Sec. 3 we introduce the spectral triple F and state the main result. Since quantum groups, coactions, etc. are defined in the framework of complex (C^* -)algebras, we replace A_F by B_F and compute the quantum isometry group of the latter in the sense of [29]. As shown in Sec. 3.2, this is given by the free product $C(U(1)) * A_{\text{aut}}(M_3(\mathbb{C}))$, where $A_{\text{aut}}(M_n(\mathbb{C}))$ is Wang's quantum automorphism group of $M_n(\mathbb{C})$ [37]. In Sec. 4, we discuss the invariance of the spectral action under quantum isometries. In Sec. 5 we explain how the result changes if we work with real instead of complex algebras. The final section deals with the proof of the main result, that is, Proposition 3.4.

Throughout the paper, by the symbol \otimes_{alg} we always mean the algebraic tensor product over \mathbb{C} , by \otimes the minimal tensor product of complex C^* -algebras or the completed tensor product of Hilbert modules over complex C^* -algebras. The symbol $\otimes_{\mathbb{R}}$ denotes the tensor product over the real numbers. Unless otherwise stated, all algebras are assumed to be unital complex associative involutive algebras. We denote by \mathcal{N}^* the set of all bounded linear functionals $\mathcal{N} \rightarrow \mathbb{C}$ on the normed linear space \mathcal{N} , by $\mathcal{M}(\mathcal{A})$ the multiplier algebra of the complex C^* -algebra \mathcal{A} , by $\mathcal{L}(\mathcal{H})$ the adjointable operators on the Hilbert module \mathcal{H} and by $\mathcal{K}(\mathcal{H})$ the compact operators on the Hilbert space \mathcal{H} . For a unital complex C^* -algebra \mathcal{A} , we implicitly use the identification of $\mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes \mathcal{A})$ with the set of all adjointable operators on the Hilbert \mathcal{A} -module $\mathcal{H} \otimes \mathcal{A}$. By abelianization of \mathcal{A} we mean the quotient of \mathcal{A} by its commutator C^* -ideal. Given a matrix u with entries u_{ij} in a C^* -algebra \mathcal{A} , we denote by $u_{ij}^* = (u_{ij})^*$ the conjugate of the element u_{ij} , and by $(u^*)_{ij} = u_{ji}^*$ the entry (i, j) of the adjoint matrix u^* . Lastly, we want to attract the reader's attention to a choice of notation. The notation $\widetilde{\text{QISO}}_J^+$ used in this article is the same as

$\widetilde{\text{QISO}}_{\text{real}}^+$ of [29]. We do this to avoid confusion with the newly defined object $\widetilde{\text{QISO}}_{\mathbb{R}}^+$ of Section 5 in the context of quantum isometries of real C^* -algebras.

2 Compact quantum groups and quantum isometries

2.1 Some generalities on Compact Quantum Groups

We begin by recalling the definition of compact quantum groups and their coactions from [40, 41]. We shall use most of the terminology of [36], for example Woronowicz C^* -subalgebra, Woronowicz C^* -ideal, etc., however with the exception that Woronowicz C^* -algebras will be called compact quantum groups, and we will not use the term compact quantum groups for the dual objects as done in [36].

Definition 2.1. A compact quantum group (to be denoted by CQG from now on) is a pair (Q, Δ) given by a complex unital C^* -algebra Q and a unital C^* -algebra morphism $\Delta : Q \rightarrow Q \otimes Q$ such that

i) Δ is coassociative, i.e.

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$$

as equality of maps $Q \rightarrow Q \otimes Q \otimes Q$;

ii) $\text{Span}\{(a \otimes 1_Q)\Delta(b) \mid a, b \in Q\}$ and $\text{Span}\{(1_Q \otimes a)\Delta(b) \mid a, b \in Q\}$ are norm-dense in $Q \otimes Q$.

For $Q = C(G)$, where G is a compact topological group, conditions i) and ii) correspond to the associativity and the cancellation property of the product in G , respectively.

Definition 2.2. A unitary corepresentation of a compact quantum group (Q, Δ) on a Hilbert space \mathcal{H} is a unitary element $U \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes Q)$ satisfying

$$(\text{id} \otimes \Delta)U = U_{(12)}U_{(13)},$$

where we use the standard leg numbering notation (see e.g. [32]).

If $Q = C(G)$, U corresponds to a strongly continuous unitary representation of G .

For any compact quantum group Q (see [40, 41]), there always exists a canonical dense $*$ -subalgebra $Q_0 \subset Q$ which is spanned by the matrix coefficients of the finite dimensional unitary corepresentations of Q and two maps $\epsilon : Q_0 \rightarrow \mathbb{C}$ (counit) and $\kappa : Q_0 \rightarrow Q_0$ (antipode) which make Q_0 a Hopf $*$ -algebra.

Definition 2.3. A Woronowicz C^* -ideal of a CQG (Q, Δ) is a C^* -ideal I of Q such that $\Delta(I) \subset \ker(\pi_I \otimes \pi_I)$, where $\pi_I : Q \rightarrow Q/I$ is the projection map. The quotient Q/I is a CQG with the induced coproduct.

If $Q = C(G)$ are continuous functions on a compact topological group G , closed subgroups of G correspond to the quotients of Q by its Woronowicz C^* -ideals. While quotients Q/I give “compact quantum subgroups”, C^* -subalgebras $Q' \subset Q$ such that $\Delta(Q') \subset Q' \otimes Q'$ describe “quotient quantum groups”.

Definition 2.4. We say that a CQG (Q, Δ) coacts on a unital C^* -algebra \mathcal{A} if there is a unital C^* -homomorphism (called a coaction) $\alpha : \mathcal{A} \rightarrow \mathcal{A} \otimes Q$ such that:

$$i) (\alpha \otimes id)\alpha = (id \otimes \Delta)\alpha,$$

$$ii) \text{Span}\{\alpha(a)(1_{\mathcal{A}} \otimes b) \mid a \in \mathcal{A}, b \in Q\} \text{ is norm-dense in } \mathcal{A} \otimes Q.$$

The coaction is faithful if any compact quantum group $Q' \subset Q$ coacting on \mathcal{A} coincides with Q .

It is well known (cf. [33, 37]) that condition (ii) in Def. 2.4 is equivalent to the existence of a norm-dense unital $*$ -subalgebra \mathcal{A}_0 of \mathcal{A} such that $\alpha(\mathcal{A}_0) \subset \mathcal{A}_0 \otimes_{\text{alg}} Q_0$ and $(id \otimes \epsilon)\alpha = id$ on \mathcal{A}_0 . For later use, let us now recall the concept of universal CQGs $A_u(R)$ as defined in [35, 38] and references therein.

Definition 2.5. For a fixed $n \times n$ positive invertible matrix R , $A_u(R)$ is the universal C^* -algebra generated by $\{u_{ij}, i, j = 1, \dots, n\}$ such that

$$uu^* = u^*u = \mathbb{I}_n, \quad u^t(R\bar{u}R^{-1}) = (R\bar{u}R^{-1})u^t = \mathbb{I}_n,$$

where $u := ((u_{ij}))$, $u^* := ((u_{ji}^*))$ and $\bar{u} := (u^*)^t = ((u_{ij}^*))$. The coproduct Δ is given by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.$$

Note that u is a unitary corepresentation of $A_u(R)$ on \mathbb{C}^n .

The $A_u(R)$'s are universal in the sense that every compact *matrix* quantum group (i.e. every CQG generated by the matrix entries of a finite-dimensional unitary corepresentation) is a quantum subgroup of $A_u(R)$ for some $R > 0$ [38]. It may also be noted that $A_u(R)$ is the universal object in the category of CQGs which admit a unitary corepresentation on \mathbb{C}^n such that the adjoint coaction on the finite-dimensional C^* -algebra $M_n(\mathbb{C})$ preserves the functional $M_n(\mathbb{C}) \ni m \mapsto \text{Tr}(R^t m)$ (see [39]).

We observe the following elementary fact which is going to be used in the sequel.

Lemma 2.6. Let $\mathcal{H} = \mathbb{C}^n$, $n \in \mathbb{N}$ and $B \in M_n(\mathcal{B})$ be a matrix with entries in a unital $*$ -algebra \mathcal{B} . Then

$$(\text{Tr}_{\mathcal{H}} \otimes id) B(L \otimes 1)B^* = \text{Tr}_{\mathcal{H}}(L) \cdot 1_{\mathcal{B}}$$

for any linear operator L on \mathcal{H} if and only if B^t is unitary.

A matrix B (with entries in a unital $*$ -algebra \mathcal{B}) such that both B and B^t are unitary is called a *biunitary* [5]. We remark that the CQG $A_u(n) := A_u(\mathbb{I}_n)$, called the *free quantum unitary group*, is generated by the biunitary matrix u given in Def. 2.5. We refer to [38] for a detailed discussion on the structure and classification of such quantum groups.

The analogue of projective unitary groups was introduced in [2] (see also Sec. 3 of [5]). Let us recall the definition.

Definition 2.7. We denote by $PA_u(n)$ the C^* -subalgebra of $A_u(n)$ generated by $\{(u_{ij})^* u_{kl} : i, j, k, l = 1, \dots, n\}$. This is a CQG with the coproduct induced from $A_u(n)$.

Remark 2.8. The projective version of any quantum subgroup of $A_u(n)$ can be defined similarly.

In [37], Wang defines the quantum automorphism group of $M_n(\mathbb{C})$, denoted by $A_{\text{aut}}(M_n(\mathbb{C}))$ to be the universal object in the category of CQGs with a coaction on $M_n(\mathbb{C})$ preserving the trace (and with morphisms given by CQGs homomorphisms intertwining the coactions). The explicit definition is in Theorem 4.1 of [37]. In the following proposition we recall Théorème 1(iv) of [2] (cf. also Prop. 3.1(3) of [5]).

Proposition 2.9 ([2, 5]). *We have $PA_u(n) \simeq A_{\text{aut}}(M_n(\mathbb{C}))$.*

Definition 2.10. *We denote by $Q_n(n')$ the amalgamated free product of n copies of $A_u(n')$ over the common Woronowicz C^* -subalgebra $PA_u(n')$. This is the CQG generated by the matrix entries of n biunitary matrices u_m ($m = 1, \dots, n$) of size n' , with relations*

$$(u_m^*)_{i,j}(u_m)_{k,l} = (u_{m'}^*)_{i,j}(u_{m'})_{k,l} \quad \forall i, j, k, l = 1, \dots, n', m, m' = 1, \dots, n,$$

and with standard matrix coproduct: $\Delta((u_m)_{ij}) = \sum_{k=1}^{n'} (u_m)_{ik} \otimes (u_m)_{kj}$ for all $m = 1, \dots, n$.

The next lemma will be needed later on.

Lemma 2.11. *Let Q be a CQG and $X, Y \in M_N(Q)$, $N \in \mathbb{N}$, be matrices with entries in Q satisfying $\Delta(X_{ik}) = \sum_{j=1}^N X_{ij} \otimes X_{jk}$ and $\Delta(Y_{ik}) = \sum_{j=1}^N Y_{ij} \otimes Y_{jk}$. Let $A \in M_N(\mathbb{C})$. Then the ideal $I \subset Q$ generated by the matrix entries of the matrix $XA - AY$ is a Woronowicz C^* -ideal.*

Proof. We now prove that $\Delta(I) \subset Q \otimes I + I \otimes Q \subset \ker(\pi_I \otimes \pi_I)$, where $\pi_I : Q \rightarrow Q/I$ is the quotient map, and hence I is a Woronowicz C^* -ideal. Since I is a (two-sided) ideal and Δ a C^* -algebra homomorphism, it is enough to give the proof for the generators $Z_{ij} := \sum_{k=1}^N (X_{ik}A_{kj} - A_{ik}Y_{kj})$ of I . The following algebraic identity holds

$$\begin{aligned} \Delta(Z_{il}) &= \sum_{j=1}^N \Delta(X_{ij}A_{jl} - A_{ij}Y_{jl}) \\ &= \sum_{j,k=1}^N X_{ij} \otimes X_{jk}A_{kl} - A_{ij}Y_{jk} \otimes Y_{kl} \\ &= \sum_{j,k=1}^N X_{ij} \otimes (X_{jk}A_{kl} - A_{jk}Y_{kl}) + (X_{ij}A_{jk} - A_{ij}Y_{jk}) \otimes Y_{kl} \\ &= \sum_{j=1}^N (X_{ij} \otimes Z_{jl} + Z_{ij} \otimes Y_{jl}). \end{aligned}$$

This concludes the proof. \square

2.2 Noncommutative Geometry and quantum isometries

In noncommutative geometry, compact Riemannian spin manifolds are replaced by real spectral triples. Recall that a *unital spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is the datum of: a complex Hilbert space \mathcal{H} , a complex unital associative involutive algebra \mathcal{A} with a faithful unital $*$ -representation $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ (the representation symbol is usually omitted), a (possibly unbounded) self-adjoint operator D on \mathcal{H} with compact resolvent and having bounded commutators with all $a \in \mathcal{A}$. The canonical commutative example is given by $(C^\infty(M), L^2(M, S), \mathcal{D})$, where $C^\infty(M)$ is the algebra of complex-valued smooth functions on a compact Riemannian spin manifold with no boundary, $L^2(M, S)$ is the Hilbert space of square integrable spinors and \mathcal{D} is the Dirac operator.

A spectral triple is *even* if there is a \mathbb{Z}_2 -grading γ on \mathcal{H} commuting with \mathcal{A} and anticommuting with D . We will set $\gamma = 1$ when the spectral triple is odd.

A spectral triple is *real* if there is an antilinear isometry $J : \mathcal{H} \rightarrow \mathcal{H}$, called the *real structure*, such that

$$J^2 = \epsilon 1, \quad JD = \epsilon' DJ, \quad J\gamma = \epsilon'' \gamma J, \quad (2.1)$$

and

$$[a, JbJ^{-1}] = 0, \quad [[D, a], JbJ^{-1}] = 0, \quad (2.2)$$

for all $a, b \in \mathcal{A}$ ¹. ϵ , ϵ' and ϵ'' are signs and determine the KO-dimension of the space [16].

For the finite part of the Standard Model $\epsilon = +1$, $\epsilon' = +1$, $\epsilon'' = -1$ and the KO-dimension is 6 [14]. Imposing a few additional conditions, it is possible to reconstruct a compact Riemannian spin manifold from any commutative real spectral triple [19].

In the example $(C^\infty(M), L^2(M, S), \not{D}, J, \gamma)$ of the spectral triple associated to a compact Riemannian spin manifold M with no boundary, there exists a covering group \tilde{G} of the group of orientation preserving isometries G of M having a unitary representation U on the Hilbert space of spinors $L^2(M, S)$ commuting with \not{D}, J, γ whose adjoint action Ad_U on $\mathcal{B}(L^2(M, S))$ preserves the subalgebra $C^\infty(M)$. This picture is used to generalize the notion of isometries as follows (cf. Def. 3 and 4 of [29]).

Definition 2.12. *A compact quantum group Q coacts by “orientation and real structure preserving isometries” on the spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ if there is a unitary corepresentation $U \in \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes Q)$ such that*

$$U \text{ commutes with } D \otimes 1 \text{ and } \gamma \otimes 1; \quad (2.3a)$$

$$(J \otimes *)U(\xi \otimes 1_Q) = U(J\xi \otimes 1_Q) \text{ for all } \xi \in \mathcal{H}; \quad (2.3b)$$

$$(\text{id} \otimes \varphi)\text{Ad}_U(a) \in \mathcal{A}'' \text{ for all } a \in \mathcal{A} \text{ and every state } \varphi \text{ on } Q, \quad (2.3c)$$

where $\text{Ad}_U = U(\cdot \otimes 1_Q)U^*$ is the adjoint coaction and \mathcal{A}'' is the double commutant of \mathcal{A} .

Note that in Definition 4 of [29] two antilinear operators J and \tilde{J} appear. \tilde{J} is a generalized real structure (it is not assumed to be an isometry) and J is its antiunitary part. As in the case of this article the real structure is an antilinear isometry J and \tilde{J} coincide and hence our definition is a particular instance of Definition 4 of [29].

We end this section by recalling Theorem 1 of [29]. Let $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ be a real spectral triple with $\epsilon' = 1$ and \mathfrak{C}_J be the category with objects (Q, U) as in Definition 2.12 and morphisms given by CQG morphisms intertwining the corresponding corepresentations. We recall that an object (Q, U) in the category \mathfrak{C}_J is said to be a sub-object of (Q_0, U_0) in the same category if there exists a CQG morphism $\varphi : Q_0 \rightarrow Q$ such that $(\text{id} \otimes \varphi)(U_0) = U$. An object (Q_0, U_0) is *universal* if for any other object (Q, U) in \mathfrak{C}_J there exists unique such φ .

Theorem 2.13 ([29]). *The category \mathfrak{C}_J has a universal object denoted by $\widetilde{\text{QISO}}^+(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ (or simply $\widetilde{\text{QISO}}_J^+(D)$) whose unitary corepresentation, say U_0 , is faithful. The quantum isometry group, denoted by $\text{QISO}^+(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ (or simply $\text{QISO}_J^+(D)$), is given by the Woronowicz C^* -subalgebra of $\widetilde{\text{QISO}}_J^+(D)$ generated by the elements $\langle \xi \otimes 1, \text{Ad}_{U_0}(a)(\eta \otimes 1) \rangle$, where $a \in \mathcal{A}$, $\xi, \eta \in \mathcal{H}$ and $\langle \cdot, \cdot \rangle$ is the $\widetilde{\text{QISO}}_J^+(D)$ -valued inner product on the Hilbert module $\mathcal{H} \otimes \widetilde{\text{QISO}}_J^+(D)$ (cf. Def. 5 in [29]).*

$\widetilde{\text{QISO}}_J^+(D)$ is the quantum analogue of the covering \tilde{G} of the classical group G of orientation preserving isometries of a spin manifold M . It's projective version (in the sense of Sec. 3 of [5]) is the quantum group $\text{QISO}_J^+(D)$, which is the quantum analogue of G .

¹Notice that in some examples, although not in the present case, the condition (2.2) has to be slightly relaxed, cf. [22, 23, 24, 25].

3 Quantum isometries of the internal non-commutative space of the Standard Model

3.1 The finite non-commutative space F

The spectral triple $(A_F, H_F, D_F, \gamma_F, J_F)$ describing the internal space F of the Standard Model is defined as follows (cf. [20] and references therein). The algebra A_F is

$$A_F := \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) , \quad (3.1)$$

where we identify \mathbb{H} with the real subalgebra of $M_2(\mathbb{C})$ with elements

$$q = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \quad (3.2)$$

for $\alpha, \beta \in \mathbb{C}$ (cf. Cayley-Dickson construction).

Let us denote by $\mathbb{C}[v_1, \dots, v_k] \simeq \mathbb{C}^k$ the vector space with basis v_1, \dots, v_k . For our convenience, we adopt the following notation for the Hilbert space H_F . It can be written as a tensor product

$$H_F := \mathbb{C}^2 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^n ,$$

where, in the notations of [20], we have

i) the first two factors $\mathbb{C}^2 \otimes \mathbb{C}^4$ with

$$\mathbb{C}^2 = \mathbb{C}[\uparrow, \downarrow] , \quad \mathbb{C}^4 = \mathbb{C}[\ell, \{q_c\}_{c=1,2,3}] ,$$

where \uparrow and \downarrow stand for weak isospin up and down, ℓ and q_c stand for lepton and quark of color c respectively. These may be combined into

$$\mathbb{C}^8 = \mathbb{C}[\nu, e, \{u_c, d_c\}_{c=1,2,3}] ,$$

where ν stands for “neutrino”, e for “electron”, u_c and d_c for quarks with weak isospin $+1/2$ and $-1/2$ respectively and of color c . Explicitly, the isomorphism $\mathbb{C}^2 \otimes \mathbb{C}^4 \rightarrow \mathbb{C}^8$ is the map

$$\uparrow \otimes \ell \mapsto \nu , \quad \downarrow \otimes \ell \mapsto e , \quad \uparrow \otimes q_c \mapsto u_c , \quad \downarrow \otimes q_c \mapsto d_c .$$

ii) a factor

$$\mathbb{C}^4 = \mathbb{C}[p_L, \bar{p}_R, \bar{p}_L, p_R] ,$$

where L, R stand for the two chiralities, p for “particle” and \bar{p} for “antiparticle”;

iii) a factor \mathbb{C}^n since each particle comes in n generations. Presently only 3 generations have been observed, but for the sake of generality we will work with an arbitrary $n \geq 3$.

From a physical point of view, rays (lines through the origin) of H_F are states describing the internal degrees of freedom of the elementary fermions. The *charge conjugation* J_F changes a particle into its antiparticle, and is the composition of the componentwise complex conjugation on H_F with the linear operator

$$J_0 := 1 \otimes 1 \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \otimes 1 . \quad (3.3)$$

The grading is

$$\gamma_F := 1 \otimes 1 \otimes \text{diag}(1, 1, -1, -1) \otimes 1.$$

The element $a = (\lambda, q, m) \in A_F$ (with $\lambda \in \mathbb{C}$, $q \in \mathbb{H}$ and $m \in M_3(\mathbb{C})$) is represented by

$$\begin{aligned} \pi(a) = & q \otimes 1 \otimes e_{11} \otimes 1 + \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \otimes 1 \otimes e_{44} \otimes 1 \\ & + 1 \otimes \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & & & \\ 0 & & m & \\ 0 & & & \end{pmatrix} \otimes (e_{22} + e_{33}) \otimes 1, \end{aligned} \quad (3.4)$$

where m is a 3×3 block and $\{e_{ij}\}_{i,j=1,\dots,k}$ is the canonical basis of $M_k(\mathbb{C})$ (e_{ij} is the matrix with 1 in the (i, j) -th position and 0 everywhere else). In particular, in (3.4) e_{11} projects on the space $\mathbb{C}[p_L]$ of particles with left chirality, e_{22} on $\mathbb{C}[\bar{p}_R]$, e_{33} on $\mathbb{C}[\bar{p}_L]$ and e_{44} on $\mathbb{C}[p_R]$.

The Dirac operator is

$$\begin{aligned} D_F := & e_{11} \otimes e_{11} \otimes \begin{pmatrix} 0 & 0 & 0 & \Upsilon_\nu \\ 0 & 0 & \Upsilon_\nu^t & \Upsilon_R \\ 0 & \bar{\Upsilon}_\nu & 0 & 0 \\ \Upsilon_\nu^* & \Upsilon_R^* & 0 & 0 \end{pmatrix} + e_{11} \otimes (1 - e_{11}) \otimes \begin{pmatrix} 0 & 0 & 0 & \Upsilon_u \\ 0 & 0 & \Upsilon_u^t & 0 \\ 0 & \bar{\Upsilon}_u & 0 & 0 \\ \Upsilon_u^* & 0 & 0 & 0 \end{pmatrix} \\ & + e_{22} \otimes e_{11} \otimes \begin{pmatrix} 0 & 0 & 0 & \Upsilon_e \\ 0 & 0 & \Upsilon_e^t & 0 \\ 0 & \bar{\Upsilon}_e & 0 & 0 \\ \Upsilon_e^* & 0 & 0 & 0 \end{pmatrix} + e_{22} \otimes (1 - e_{11}) \otimes \begin{pmatrix} 0 & 0 & 0 & \Upsilon_d \\ 0 & 0 & \Upsilon_d^t & 0 \\ 0 & \bar{\Upsilon}_d & 0 & 0 \\ \Upsilon_d^* & 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (3.5)$$

where each of the Υ matrices are in $M_n(\mathbb{C})$, $\bar{m} := (m^*)^t$ is the matrix obtained from m by conjugating each entry, and we identify $\mathcal{B}(H_F) = M_2(\mathbb{C}) \otimes M_4(\mathbb{C}) \otimes (M_4(\mathbb{C}) \otimes M_n(\mathbb{C}))$ with $M_2(\mathbb{C}) \otimes M_4(\mathbb{C}) \otimes M_{4n}(\mathbb{C})$ by writing $M_{4n}(\mathbb{C})$ as a 4×4 matrix with entries in $M_n(\mathbb{C})$; in particular $e_{ij} \otimes m \in M_4(\mathbb{C}) \otimes M_n(\mathbb{C})$ will be the matrix with the $n \times n$ block m in position (i, j) .

The matrix Υ_R is symmetric, the other Υ matrices are positive. Their physical meaning is explained in section 17.4 of [20]: for $x = e, u, d$ the eigenvalues of $\Upsilon_x^* \Upsilon_x$ give the square of the masses of the n generations of the particle x ; the eigenvalues of $\Upsilon_\nu^* \Upsilon_\nu$ give the Dirac masses of neutrinos; the eigenvalues of $\Upsilon_R^* \Upsilon_R$ give the Majorana masses of neutrinos.

If we replace a spectral triple with one that is unitary equivalent we do not change the symmetries. From Theorem 1.187(3) (and analogously to Lemma 1.190) of [20] it follows that, modulo an unitary equivalence, we can diagonalize one element of each pair $(\Upsilon_\nu, \Upsilon_e)$ and (Υ_u, Υ_d) . We choose to diagonalize Υ_u and Υ_e .

Thus, we make the following hypothesis on the Υ matrices:

- Υ_u and Υ_e are positive, diagonal and their eigenvalues are non-zero.
- Υ_d and Υ_ν are positive, the eigenvalues of Υ_d are non-zero. Let us denote by C the $SU(n)$ matrices such that $\Upsilon_d = C \delta_\downarrow C^*$, where δ_\downarrow is a diagonal matrix with non-negative eigenvalues. C is the so-called Cabibbo-Kobayashi-Maskawa matrix, responsible for the quark mixing, cf. Sec. 9.3 of [20]. Similarly the unitary diagonalizing Υ_ν is the so-called Pontecorvo-Maki-Nakagawa-Sakata matrix, responsible for the neutrino mixing, cf. Sec. 9.6 of [20].

- Υ_R is symmetric.
- For physical reasons, we assume that: Υ_x and Υ_y have distinct eigenvalues, for all $x, y \in \{\nu, e, u, d\}$ with $x \neq y$; eigenvalues of Υ_e , Υ_u and Υ_d are non-zero and with multiplicity one.

Remark 3.1. We will often use the fact that $\Upsilon^t = \overline{\Upsilon}$ for any positive matrix Υ .

3.2 Quantum isometries of F

Since the definition of quantum isometry group is given for spectral triples over complex $*$ -algebras, we first need to explain how to canonically associate one to any spectral triple over a real $*$ -algebra.

Lemma 3.2. To any real spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ over a real $*$ -algebra \mathcal{A} we can associate a real spectral triple $(\mathcal{B}, \mathcal{H}, D, \gamma, J)$ over the complex $*$ -algebra $\mathcal{B} \simeq \mathcal{A}_{\mathbb{C}} / \ker \pi_{\mathbb{C}}$, where $\mathcal{A}_{\mathbb{C}} \simeq \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of \mathcal{A} , with conjugation defined by $(a \otimes_{\mathbb{R}} z)^* = a^* \otimes_{\mathbb{R}} \overline{z}$ for $a \in \mathcal{A}$ and $z \in \mathbb{C}$, and $\pi_{\mathbb{C}} : \mathcal{A}_{\mathbb{C}} \rightarrow \mathcal{B}(\mathcal{H})$ is the $*$ -representation

$$\pi_{\mathbb{C}}(a \otimes_{\mathbb{R}} z) = z\pi(a), \quad a \in \mathcal{A}, z \in \mathbb{C}. \quad (3.6)$$

Notice that $\ker \pi_{\mathbb{C}}$ may be nontrivial since the representation $\pi_{\mathbb{C}}$ is not always faithful. For example, if \mathcal{A} is itself a complex $*$ -algebra (every complex $*$ -algebra is also a real $*$ -algebra) and π is complex linear, then for any $a \in \mathcal{A}$ the element $a \otimes_{\mathbb{R}} 1 + ia \otimes_{\mathbb{R}} i$ of $\mathcal{A}_{\mathbb{C}}$ is in the kernel of $\pi_{\mathbb{C}}$. This happens in the Standard Model case, where the complexification of $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ is the algebra $(A_F)_{\mathbb{C}} := \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C})$, where we have used the complex $*$ -algebra isomorphism $M_n(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} \rightarrow M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ given by

$$m \otimes_{\mathbb{R}} z \mapsto (mz, \overline{m}z)$$

having inverse

$$(m, m') \mapsto \frac{m + \overline{m}'}{2} \otimes_{\mathbb{R}} 1 + \frac{m - \overline{m}'}{2i} \otimes_{\mathbb{R}} i \quad (3.7)$$

for all $m, m' \in M_n(\mathbb{C})$, $z \in \mathbb{C}$.

Using (3.6), (3.7) and (3.4) we get $\pi_{\mathbb{C}}(\lambda, \lambda', q, m, m') = \langle \lambda, \lambda', q, m \rangle$, where

$$\begin{aligned} \langle \lambda, \lambda', q, m \rangle &:= q \otimes 1 \otimes e_{11} \otimes 1 + \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix} \otimes 1 \otimes e_{44} \otimes 1 \\ &\quad + 1 \otimes \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & & & \\ 0 & & m & \\ 0 & & & \end{pmatrix} \otimes (e_{22} + e_{33}) \otimes 1. \end{aligned} \quad (3.8)$$

The complex $*$ -algebra $B_F := (A_F)_{\mathbb{C}} / \ker \pi_{\mathbb{C}}$ is simply the algebra $B_F \simeq \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ with elements $\langle \lambda, \lambda', q, m \rangle$. With A_F replaced by B_F , we can now study quantum isometries.

We notice that in the case of the spectral triple of the internal part of the Standard Model, the conditions (2.3b-2.3c) are equivalent to

$$(J_0 \otimes 1)\overline{U} = U(J_0 \otimes 1); \quad (3.9a)$$

$$\text{Ad}_U(B_F) \subset B_F \otimes_{\text{alg}} Q; \quad (3.9b)$$

with J_0 given by (3.3). The equivalence between (2.3b) and (3.9a) is an immediate consequence of the definition of J_F . The equivalence between (2.3c) and (3.9b) follows from the equality of B_F'' and B_F , since the latter is a finite-dimensional C^* -algebra.

We need a preparatory lemma before our main proposition.

Lemma 3.3. *Let Q be the universal C^* -algebra generated by unitary elements x_k ($k = 0, \dots, n$), the matrix entries of 3×3 biunitaries T_m ($m = 1, \dots, n$) and of an $n \times n$ biunitary V , with relations*

$$\text{diag}(x_0 x_1, \dots, x_0 x_n) \Upsilon_\nu = \Upsilon_\nu \text{diag}(x_0 x_1, \dots, x_0 x_n) = \bar{V} \Upsilon_\nu = \Upsilon_\nu \bar{V}, \quad V \Upsilon_R = \Upsilon_R \bar{V}, \quad (3.10a)$$

$$\sum_{m=1}^n C_{rm} \bar{C}_{sm} (T_m)_{j,k} = 0, \quad \forall r \neq s, \quad (r, s = 1, \dots, n; j, k = 1, 2, 3) \quad (3.10b)$$

$$(T_m^*)_{i,j} (T_m)_{k,l} = (T_{m'}^*)_{i,j} (T_{m'})_{k,l}, \quad \forall m, m', \quad (i, j, k, l = 1, 2, 3, m, m' = 1, \dots, n) \quad (3.10c)$$

where $C = ((C_{r,s}))$ is the CKM matrix. Then Q with matrix coproduct

$$\Delta(x_k) = x_k \otimes x_k, \quad \Delta((T_m)_{ij}) = \sum_{l=1,2,3} (T_m)_{il} \otimes (T_m)_{lj}, \quad \Delta(V_{ij}) = \sum_{l=1,\dots,n} V_{il} \otimes V_{lj}, \quad (3.11)$$

is a quantum subgroup of the free product

$$\underbrace{C(U(1)) * C(U(1)) * \dots * C(U(1))}_{n+1} * Q_n(3) * A_u(n). \quad (3.12)$$

The Woronowicz C^* -ideal of (3.12) defining Q is determined by the relations (3.10a) and (3.10b).

Proof. $Q_n(3)$ is by definition generated by 3×3 biunitaries T'_m ($m = 1, \dots, n$) with the relation (3.10c), $A_u(n)$ is generated by the matrix entries of a $n \times n$ biunitary V' , and $C(U(1)) * C(U(1)) * \dots * C(U(1))$ is freely generated by unitary elements x'_k ($k = 0, \dots, n$). The map $T'_m \mapsto T_m$, $V' \mapsto V$ and $x'_k \mapsto x_k$ defines a surjective C^* -algebra morphism from the CQG in (3.12) to Q .

From Lemma 2.11, it follows that the kernel of the morphism $(V', x'_k) \mapsto (V, x_k)$ is a Woronowicz C^* -ideal, i.e. the relations (3.10a) define a quantum subgroup of $C(U(1)) * C(U(1)) * \dots * C(U(1)) * A_u(n)$ (apply the Lemma to $A = \Upsilon_\nu$ and $X, Y \in \{\text{diag}(x_0 x_1, \dots, x_0 x_n), V\}$).

It remains to prove that the kernel I of the morphism $T'_m \mapsto T_m$ is also a Woronowicz C^* -ideal, i.e. the quotient of $Q_n(3)$ by the relation (3.10b) is a CQG. The ideal I is generated by the elements $X_{r,s,j,k} := \sum_{m=1}^n C_{rm} \bar{C}_{sm} (T'_m)_{j,k}$ for all $j, k = 1, 2, 3$, $r, s = 1, \dots, n$ and $r \neq s$. An easy computation shows that

$$\begin{aligned} \Delta(X_{r,s,j,k}) &= \sum_{m=1}^n \sum_{l=1}^3 C_{rm} \bar{C}_{sm} (T'_m)_{j,l} \otimes (T'_m)_{l,k} \\ &= \sum_{l,p=1}^3 \sum_{m=1}^n C_{rm} \bar{C}_{pm} (T'_m)_{j,l} \otimes \sum_{m'=1}^n C_{pm'} \bar{C}_{sm'} (T'_{m'})_{l,k} \\ &= \sum_{l,p=1}^3 X_{r,p,j,l} \otimes X_{p,s,l,k}, \end{aligned}$$

where the second equality follows from $\sum_{p=1}^3 \bar{C}_{pm} C_{pm'} = (C^* C)_{mm'} = \delta_{mm'}$ (recall that C is a unitary matrix). Hence $\Delta(I) \subset I \otimes I$, so that I is a Woronowicz C^* -ideal. This concludes the proof. \square

Proposition 3.4. *The universal object $\widetilde{\text{QISO}}_J^+(D_F)$ of the category \mathfrak{C}_J is given by the CQG in Lemma 3.3 with corepresentation*

$$\begin{aligned}
U = & e_{11} \otimes e_{11} \otimes e_{11} \otimes \sum_{k=1}^n e_{kk} \otimes x_0 x_k + e_{22} \otimes e_{11} \otimes (e_{11} + e_{44}) \otimes \sum_{k=1}^n e_{kk} \otimes x_k \\
& + e_{11} \otimes e_{11} \otimes e_{33} \otimes \sum_{k=1}^n e_{kk} \otimes x_k^* x_0^* + e_{22} \otimes e_{11} \otimes (e_{22} + e_{33}) \otimes \sum_{k=1}^n e_{kk} \otimes x_k^* \\
& + e_{11} \otimes e_{11} \otimes e_{22} \otimes \sum_{j,k=1}^n e_{jk} \otimes (V)_{jk} + e_{11} \otimes e_{11} \otimes e_{44} \otimes \sum_{j,k=1}^n e_{jk} \otimes (\bar{V})_{jk} \\
& + e_{11} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes (e_{11} + e_{44}) \otimes \sum_{m=1}^n e_{mm} \otimes (T_m)_{j,k} \\
& + e_{22} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes (e_{11} + e_{44}) \otimes \sum_{m=1}^n e_{mm} \otimes x_0^* (T_m)_{j,k} \\
& + e_{11} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes (e_{22} + e_{33}) \otimes \sum_{m=1}^n e_{mm} \otimes (\bar{T}_m)_{j,k} \\
& + e_{22} \otimes \sum_{j,k=1,2,3} e_{j+1,k+1} \otimes (e_{22} + e_{33}) \otimes \sum_{m=1}^n e_{mm} \otimes (\bar{T}_m)_{j,k} x_0 . \tag{3.13}
\end{aligned}$$

$\widetilde{\text{QISO}}_J^+(D_F)$ coacts trivially on the two summands \mathbb{C} of $B_F = \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$, while on the remaining summands the coaction is

$$\alpha(\langle 0, 0, e_{ii}, 0 \rangle) = \langle 0, 0, e_{ii}, 0 \rangle \otimes 1 , \tag{3.14a}$$

$$\alpha(\langle 0, 0, e_{12}, 0 \rangle) = \langle 0, 0, e_{12}, 0 \rangle \otimes x_0 , \tag{3.14b}$$

$$\alpha(\langle 0, 0, e_{21}, 0 \rangle) = \langle 0, 0, e_{21}, 0 \rangle \otimes x_0^* , \tag{3.14c}$$

$$\alpha(\langle 0, 0, 0, e_{ij} \rangle) = \sum_{k,l=1,2,3} \langle 0, 0, 0, e_{kl} \rangle \otimes (T_1^*)_{i,k} (T_1)_{l,j} . \tag{3.14d}$$

Proof. The proof is in Sec. 6. □

Definition 3.5. Let $Q_{n,C}(3)$ be the quantum subgroup of $Q_n(3)$, cf. Def. 2.10, defined by the relation $\sum_{m=1}^n C_{rm} \bar{C}_{sm} (u_m)_{j,k} = 0$.

Remark 3.6. It is easy to see that $Q_{n,C}(3)$ is noncommutative as a C^* -algebra. Indeed, if u is a 3×3 biunitary generating $A_u(3)$, the map

$$(u_m)_{jk} \mapsto u_{jk} , \quad \forall m = 1, \dots, n , \quad j, k = 1, 2, 3 ,$$

is a C^* -algebra morphism (C is a unitary matrix, hence (3.10b) and (3.10c) are automatically satisfied). Thus $A_u(3)$ is a quantum subgroup of $Q_{n,C}(3)$.

Proposition 3.7. The quantum isometry group of the internal space of the Standard Model is

$$\text{QISO}_J^+(D_F) = C(U(1)) * A_{\text{aut}}(M_3(\mathbb{C})) .$$

Its abelianization is given by (complex functions on) the classical group $U(1) \times PU(3)$.

Proof. From (3.14) it follows that $\text{QISO}_J^+(D_F)$ is generated by x_0 and $(T_1^*)_{i,k}(T_1)_{l,j}$: then $\text{QISO}_J^+(D_F)$ is a quantum subgroup of $C(U(1)) * PA_u(3)$. On the other hand $A_u(3)$ is a quantum subgroup of $Q_{n,C}(3)$ (Rem. 3.6), and with the map $x_0 \mapsto x_0$, $x_i \mapsto 1$ ($i = 1, \dots, n$), $V \mapsto x_0 1_n$ and $(T_m)_{jk} \mapsto u_{jk} \forall m = 1, \dots, n$ (with u_{jk} the usual generators of $A_u(3)$), one proves that $C(U(1)) * A_u(3)$ is a sub-object of $\widetilde{\text{QISO}}_J^+(D_F)$ in the category \mathfrak{C}_J and $C(U(1)) * PA_u(3)$ is a quantum subgroup of $\text{QISO}_J^+(D_F)$; hence $\text{QISO}_J^+(D_F)$ and $C(U(1)) * PA_u(3)$ coincide. Recalling that $PA_u(3) \simeq A_{\text{aut}}(M_3(\mathbb{C}))$ (cf. Def. 2.7 and Prop. 2.9) the proof is concluded. \square

Although $\widetilde{\text{QISO}}_J^+(D_F)$ depends on Υ_ν , Υ_R and the CKM matrix C (cf. (3.10)), the quantum group $\text{QISO}_J^+(D_F)$ does not depend on the explicit form of these two matrices. We stress the importance of this results, since neutrino masses are not known (at the moment, we only know that they are all distinct [27, 1]). Also, $\text{QISO}_J^+(D_F)$ is independent on the number of generations.

Let us conclude this section by explaining how elementary particles transform under the corepresentation U in physics notation. As explained in Sec. 3.1, we have

$$\begin{aligned} \nu_{L,k} &:= e_1 \otimes e_1 \otimes e_1 \otimes e_k, & (\text{left-handed neutrino, generation } k) \\ \nu_{R,k} &:= e_1 \otimes e_1 \otimes e_4 \otimes e_k, & (\text{right-handed neutrino, generation } k) \\ e_{L,k} &:= e_2 \otimes e_1 \otimes e_1 \otimes e_k, & (\text{left-handed electron, generation } k) \\ e_{R,k} &:= e_2 \otimes e_1 \otimes e_4 \otimes e_k, & (\text{right-handed electron, generation } k) \\ u_{L,c,k} &:= e_1 \otimes e_{c+1} \otimes e_1 \otimes e_k, & (\text{left-handed up-quark, color } c, \text{ generation } k) \\ u_{R,c,k} &:= e_1 \otimes e_{c+1} \otimes e_4 \otimes e_k, & (\text{right-handed up-quark, color } c, \text{ generation } k) \\ d_{L,c,k} &:= e_2 \otimes e_{c+1} \otimes e_1 \otimes e_k, & (\text{left-handed down-quark, color } c, \text{ generation } k) \\ d_{R,c,k} &:= e_2 \otimes e_{c+1} \otimes e_4 \otimes e_k, & (\text{right-handed down-quark, color } c, \text{ generation } k) \end{aligned}$$

where $\{e_i, i = 1, \dots, r\}$ is the canonical orthonormal basis of \mathbb{C}^r , $c = 1, 2, 3$ and $k = 1, \dots, n$. These together with the corresponding antiparticles form a linear basis of H_F . A straightforward computation using (3.13) proves that we have the following transformation laws

$$\begin{aligned} U(\nu_{L,k}) &:= \nu_{L,k} \otimes x_0 x_k, & U(\nu_{R,k}) &:= \sum_{j=1}^n \nu_{R,j} \otimes \bar{V}_{jk}, \\ U(e_{L,k}) &:= e_{L,k} \otimes x_k, & U(e_{R,k}) &:= e_{R,k} \otimes x_k, \\ U(u_{L,c,k}) &:= \sum_{c'=1}^3 u_{L,c',k} \otimes (T_k)_{c'c}, & U(u_{R,c,k}) &:= \sum_{c'=1}^3 u_{R,c',k} \otimes (T_k)_{c'c}, \\ U(d_{L,c,k}) &:= \sum_{c'=1}^3 d_{L,c',k} \otimes x_0^* (T_k)_{c'c}, & U(d_{R,c,k}) &:= \sum_{c'=1}^3 d_{R,c',k} \otimes x_0^* (T_k)_{c'c}, \end{aligned}$$

where $U(v)$, $v \in H_F$, is a shorthand notation for $U(v \otimes 1_Q)$. Antiparticles transform according to the conjugate corepresentations.

We comment now on the meaning of $\widetilde{\text{QISO}}_J^+(D_F)$.

Remark 3.8. Let z be the generator of $C(U(1))$, $T = ((T_{jk}))$ be the generators of $A_u(3)$ and consider the corepresentation $H_F \rightarrow H_F \otimes (C(U(1)) * A_u(3))$ determined by

$$\nu_{\bullet,k} \mapsto \nu_{\bullet,k} \otimes 1, \quad e_{\bullet,k} \mapsto e_{\bullet,k} \otimes (z^*)^3, \quad (3.15a)$$

$$u_{\bullet,c,k} \mapsto \sum_{c'=1}^3 u_{\bullet,c',k} \otimes z^2 T_{c'c}, \quad d_{\bullet,c,k} \mapsto \sum_{c'=1}^3 d_{\bullet,c',k} \otimes z^* T_{c'c}, \quad (3.15b)$$

where \bullet is L or R . Let q be a third root of unity, and consider the \mathbb{Z}_3 action on $C(U(1)) * A_u(3)$ given by $z \mapsto qz$, $T_{jk} \mapsto qT_{jk}$. The elements appearing in the image of the above corepresentation generate the fixed point subalgebra for this action, that is $\{C(U(1)) * A_u(3)\}^{\mathbb{Z}_3}$.

The quantum group $\{C(U(1)) * A_u(3)\}^{\mathbb{Z}_3}$ with the corepresentation above is a sub-object of $\widetilde{\text{QISO}}_J^+(D_F)$ in the category \mathfrak{E}_J . The surjective CQG homomorphism $\widetilde{\text{QISO}}_J^+(D_F) \rightarrow \{C(U(1)) * A_u(3)\}^{\mathbb{Z}_3}$ is given by

$$x_0 \mapsto z^3, \quad x_m \mapsto (z^*)^3, \quad \forall m = 1, \dots, n, \quad (T_m)_{jk} \mapsto z^2 T_{jk}, \quad \forall m = 1, \dots, n, \quad V \mapsto 1_n.$$

The kernel of this map — the ideal generated by V_{jk} and by products $x_0 x_k$ and $(T_m^* T_{m'})_{kl}$ for all $m \neq m'$ — is given by elements that do not appear in the adjoint coaction on B_F . Roughly speaking, modulo terms “commuting” with the algebra B_F , we have that $\widetilde{\text{QISO}}_J^+(D_F) \sim \{C(U(1)) * A_u(3)\}^{\mathbb{Z}_3}$ is the “free version” of the ordinary gauge group after symmetry breaking.

If we pass to the abelianization $C(U(1) \times U(3))^{\mathbb{Z}_3} \simeq C((U(1) \times U(3))/\mathbb{Z}_3)$ of $\{C(U(1)) * A_u(3)\}^{\mathbb{Z}_3}$ and from the corresponding corepresentation to the dual representation of $(\tau, g) \in U(1) \times U(3)$, from (3.15) we find the usual global gauge transformations after symmetry breaking:

$$\nu_{\bullet,k} \mapsto \nu_{\bullet,k}, \quad e_{\bullet,k} \mapsto (\tau^*)^3 e_{\bullet,k}, \quad u_{\bullet,c,k} \mapsto \sum_{c'=1}^3 \tau^2 g_{c'c} u_{\bullet,c',k}, \quad d_{\bullet,c,k} \mapsto \sum_{c'=1}^3 \tau^* g_{c'c} d_{\bullet,c',k}.$$

3.3 $\widetilde{\text{QISO}}_J^+$ in two special cases

As we already noticed, $\widetilde{\text{QISO}}_J^+(D_F)$ depends upon the explicit form of Υ_ν , Υ_R and C . In particular, on one extreme we have the case when Υ_ν is invertible (this is the case of the Dirac operator in the moduli space as in Prop. 1.192 of [20]) and on the other extreme we have the case $\Upsilon_\nu = 0$.

Proposition 3.9. *If Υ_ν is invertible, $\widetilde{\text{QISO}}_J^+(D_F)$ is the free product of $Q_{n,C}(3)$ with the quotient of*

$$\underbrace{C(U(1)) * C(U(1)) * \dots * C(U(1))}_{n+1}$$

by the relations

$$\begin{aligned} x_i^* x_0^* &= x_0 x_j & \forall i, j \text{ such that } (\Upsilon_R)_{ij} \neq 0, \\ x_i &= x_j & \forall i, j \text{ such that } (\Upsilon_\nu)_{ij} \neq 0. \end{aligned}$$

Proof. If Υ_ν is invertible, the first equation in (3.10a) gives $V = \text{diag}(x_1^* x_0^*, \dots, x_n^* x_0^*)$ (so that the factor $A_u(n)$ in (3.12) disappears) and also $(\Upsilon_\nu)_{ij} x_0(x_i - x_j) = 0$. The latter implies $x_i = x_j$ whenever $(\Upsilon_\nu)_{ij} \neq 0$.

The second equation in (3.10a) becomes $(\Upsilon_R)_{ij}(x_i^* x_0^* - x_0 x_j) = 0$, which implies $x_i^* x_0^* = x_0 x_j$ whenever $(\Upsilon_R)_{ij} \neq 0$. \square

Although disproved by experiment, it is an interesting exercise to study the case of massless ($\Upsilon_\nu = 0$) left-handed neutrinos, that is the so-called *minimal* Standard Model.

Proposition 3.10. *If $\Upsilon_\nu = 0$, $\widetilde{\text{QISO}}_J^+(D_F)$ is isomorphic to*

$$\underbrace{C(U(1)) * C(U(1)) * \dots * C(U(1))}_{n+1} * Q_{n,C}(3) * A',$$

where $A' := A_u(n)/\sim$, $A_u(n)$ is generated by the $n \times n$ biunitary V and “ \sim ” is the relation $V\Upsilon_R = \Upsilon_R\bar{V}$.

As a consequence of Noether's theorem, any Lie group symmetry is associated to a corresponding conservation law. We shall see in Sec. 4.2 that $\widetilde{\text{QISO}}_J^+(D_F)$ is indeed a symmetry of the dynamics. In Rem. 3.8, roughly speaking, we discussed the part of $\widetilde{\text{QISO}}_J^+$ that is relevant in the coaction on the algebra B_F : it is the free version of the gauge group which corresponds to the conservation of color and electric charge. We complete here the analysis by discussing the additional symmetries that are present in the case of the minimal Standard Model in Prop. 3.10.

The factor A' coacts only on the subspace $(e_{11} \otimes e_{11} \otimes (e_{22} + e_{44}) \otimes 1)H_F$ of right-handed neutrinos, and can be neglected in the minimal Standard Model (where we consider only left-handed neutrinos). As a consequence of Noether's theorem, there exists a conservation law corresponding to each classical group of symmetries.

It is easy to give an interpretation to the $C(U(1))$ factors generated by x_i , $i = 1, \dots, n$. Passing from the $C(U(1))$ coaction to the dual $U(1)$ action, one easily sees that for $i > 0$, x_i gives a phase transformation of the i -th generation of ν_L, e_L, e_R (plus the opposite transformation for the antiparticles). In the minimal Standard Model, which has only left-handed (massless) neutrinos, these symmetries give the conservation laws of the total number of leptons in each generation (electron number, muon number, tau number, plus other $n - 3$ for the other families of leptons).

To conclude the list of conservation laws, there is still one classical $U(1)$ subgroup of the factor $Q_{n,C}(3)$ that should be mentioned. If we denote by y the unitary generator of $C(U(1))$, a surjective CQG homomorphism $\varphi : \widetilde{\text{QISO}}_J^+(D_F) \rightarrow C(U(1))$ is given by

$$x_0 \mapsto 1, \quad x_i \mapsto 1, \quad V_{j,k} \mapsto \delta_{j,k}, \quad (T_i)_{j,k} \mapsto \delta_{j,k} y,$$

for all $i = 1, \dots, n$ and $j, k = 1, 2, 3$. From U we get the following corepresentation of this $U(1)$ subgroup on H_F :

$$\begin{aligned} (id \otimes \varphi)(U) &= 1 \otimes e_{11} \otimes 1 \otimes 1 \otimes 1_{C(U(1))} \\ &\quad + 1 \otimes (1 - e_{11}) \otimes (e_{11} + e_{44}) \otimes 1 \otimes y \\ &\quad + 1 \otimes (1 - e_{11}) \otimes (e_{22} + e_{33}) \otimes 1 \otimes y^*. \end{aligned}$$

The representation of $U(1)$ dual to this corepresentation of $C(U(1))$ is given by a phase transformation on the subspace $\mathbb{C}^2 \otimes (1 - e_{11})\mathbb{C}^4 \otimes (e_{11} + e_{44})\mathbb{C}^4 \otimes \mathbb{C}^n$ of quarks and the inverse transformation on the subspace $\mathbb{C}^2 \otimes (1 - e_{11})\mathbb{C}^4 \otimes (e_{22} + e_{33})\mathbb{C}^4 \otimes \mathbb{C}^n$ of anti-quarks and is called in physics the “baryon phase symmetry”. It corresponds to the conservation of the baryon number (total number of quarks minus the number of anti-quarks).

In this section we discussed conservation laws associated to classical subgroups of $\widetilde{\text{QISO}}_J^+(D_F)$ in the massless neutrino case. It would be interesting to extend this study to the full quantum group $\widetilde{\text{QISO}}_J^+(D_F)$ in the sense of a suitable Noether analysis extended to the quantum group framework. If we consider massive neutrinos, we lose a lot of classical symmetries, but we still have many quantum symmetries. A natural question is whether quantum symmetries are suitable for deriving conservation laws (i.e. physical predictions). A first step in this direction is to investigate whether the spectral action is invariant under quantum isometries. We discuss this point in the next section.

4 Quantum isometries of $M \times F$

4.1 Quantum isometries of a product of spectral triples

Before discussing the spectral action, we want to understand whether the quantum isometry group of the finite geometry F is also a quantum group of orientation preserving isometries of the full spectral triple of the Standard Model, that is the product of F with the canonical spectral triple of a compact Riemannian spin manifold M with no boundary. The answer is affirmative and we can prove it in a more general situation:

- Let $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1, J_1)$ be any unital real spectral triple ($\gamma_1 = 1$ if the spectral triple is odd).
- Let $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2, J_2)$ be a finite-dimensional unital even real spectral triple.
- Let $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ be the product triple, i.e.

$$\begin{aligned} \mathcal{A} &:= \mathcal{A}_1 \otimes_{\text{alg}} \mathcal{A}_2, & \mathcal{H} &:= \mathcal{H}_1 \otimes \mathcal{H}_2, & D &:= D_1 \otimes \gamma_2 + 1 \otimes D_2, \\ \gamma &:= \gamma_1 \otimes \gamma_2, & J &:= J_1 \otimes J_2. \end{aligned}$$

In the case of the Standard Model, $(\mathcal{A}_1, \mathcal{H}_1, D_1, \gamma_1, J_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2, J_2)$ will be the canonical spectral triple of M and the spectral triple $(B_F, H_F, D_F, \gamma_F, J_F)$ respectively.

We claim that:

Lemma 4.1. $\widetilde{\text{QISO}}^+(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2, J_2)$ coacts by “orientation and real structure preserving isometries” on the product triple $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$.

Proof. Let Q_0 be the quantum group $\widetilde{\text{QISO}}_{J_2}^+(D_2)$ and U its corepresentation on \mathcal{H}_2 . Then $\hat{U} := 1 \otimes U$ is a unitary corepresentation on $\mathcal{H}_1 \otimes \mathcal{H}_2$, and we need to prove that it satisfies (2.3a), (2.3b), and (2.3c). The first two conditions are easy to check. Indeed, if U commutes with D_2 and γ_2 , clearly $1 \otimes U$ commutes with $D = D_1 \otimes \gamma_2 + 1 \otimes D_2$ and $\gamma = \gamma_1 \otimes \gamma_2$. Moreover, for any vector $\xi = \xi_1 \otimes \xi_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$,

$$\begin{aligned} (J \otimes *)\hat{U}(\xi \otimes 1) &= (J_1 \otimes J_2 \otimes *) (1 \otimes U)(\xi_1 \otimes \xi_2 \otimes 1) \\ &= J_1 \xi_1 \otimes (J_2 \otimes *)U(\xi_2 \otimes 1) \\ &= J_1 \xi_1 \otimes U(J_2 \xi_2 \otimes 1) \\ &= (1 \otimes U)(J_1 \xi_1 \otimes J_2 \xi_2 \otimes 1) \\ &= \hat{U}(J\xi \otimes 1), \end{aligned}$$

and thus (2.3b) is proved.

Any element of \mathcal{A} is a finite sum of tensors $a_1 \otimes a_2$, with $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$, and since \mathcal{A}_2 is finite dimensional implies $U(a_2 \otimes 1_{Q_0})U^* \in \mathcal{A}_2 \otimes_{\text{alg}} Q_0$, we have

$$\text{Ad}_{\hat{U}}(a_1 \otimes a_2) = \hat{U}(a_1 \otimes a_2 \otimes 1_{Q_0})\hat{U}^* = a_1 \otimes U(a_2 \otimes 1_{Q_0})U^* \in \mathcal{A}_1 \otimes_{\text{alg}} \mathcal{A}_2 \otimes_{\text{alg}} Q_0$$

which implies (2.3c). □

4.2 Invariance of the spectral action

The dynamics of a unital spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$ — with $\gamma = 1$ in the odd case — is governed by an action functional [12]

$$S[A, \psi] := S_b[A] + S_f[A, \psi] ,$$

whose variables are a self-adjoint one-form $A \in \Omega_D^{1,s,a} \subset \mathcal{B}(\mathcal{H})$ and ψ either in \mathcal{H} or in $\mathcal{H}_+ := (1 + \gamma)\mathcal{H}$. While one uses \mathcal{H} in Yang-Mills theories, the reduction to \mathcal{H}_+ is employed in the Standard Model to solve the fermion doubling problem [31, 20]. The fermionic part of the spectral action is either

$$S_f[A, \psi] = \langle \psi, D_A \psi \rangle , \quad D_A := D + A , \quad (4.1)$$

or for a real spectral triple

$$S_f[A, \psi] := \langle J\psi, D_A \psi \rangle , \quad D_A := D + A + \epsilon' J A J^{-1} , \quad (4.2)$$

where ϵ' is the sign in (2.1). The bosonic part is

$$S_b[A] = \text{Tr } f(D_A/\Lambda) ,$$

where D_A is either the operator in (4.1) or (4.2), and f is a suitable cut-off function (with $\Lambda > 0$). More precisely, f is a smooth approximation of the characteristic function of the interval $[-1, 1]$, so that $f(D_A/\Lambda)$ — defined via the continuous functional calculus — is a trace class operator on \mathcal{H} and $S_b[A]$ is well defined.

In the rest of the section we focus on the fermionic action S_f and the operator D_A given by (4.2), although all the proofs can be repeated in the case (4.1) as well.

Assume that Q is a CQG with a unitary corepresentation \hat{U} on \mathcal{H} commuting with D and γ , and such that $\text{Ad}_{\hat{U}}$ maps \mathcal{A} into $\mathcal{A} \otimes_{\text{alg}} Q$ (rather than (2.3c)). Then \mathcal{H}_+ is preserved by \hat{U} , and for any 1-form $A = \sum_i a_i [D, b_i]$, with $a_i, b_i \in \mathcal{A}$, the operator $\text{Ad}_{\hat{U}}(A) = \hat{U}(A \otimes 1)\hat{U}^* = \sum_i \text{Ad}_{\hat{U}}(a_i)[D \otimes 1, \text{Ad}_{\hat{U}}(b_i)]$ is an element of $\Omega_D^1 \otimes_{\text{alg}} Q$. Therefore a coaction of Q on $\Omega_D^1 \oplus \mathcal{H}_+$ is given by

$$\beta : (A, \psi) \mapsto (\hat{U}(A \otimes 1)\hat{U}^*, \hat{U}(\psi \otimes 1)) .$$

To discuss the (co)invariance of the spectral action we need to extend it to the latter space. There is a natural way to do it. The inner product $\langle \cdot, \cdot \rangle : \mathcal{H}_+ \otimes \mathcal{H}_+ \rightarrow \mathbb{C}$ can be extended in a unique way to an Hermitian structure $\langle \cdot, \cdot \rangle_Q : \mathcal{M} \otimes \mathcal{M} \rightarrow Q$ on the right Q -module $\mathcal{M} := \mathcal{H}_+ \otimes Q$ by the rule $\langle \psi \otimes q, \psi' \otimes q' \rangle_Q = q^* q' \langle \psi, \psi' \rangle$. Unitary (resp. antiunitary) maps L on \mathcal{H}_+ are extended in a unique way to Q -linear (resp. antilinear) maps on \mathcal{M} as $L \otimes 1$ (resp. $L \otimes *$). The corresponding extension of the spectral action is given by the Q -valued functional

$$\tilde{S}[\tilde{A}, \tilde{\psi}] := \tilde{S}_b[\tilde{A}] + \tilde{S}_f[\tilde{A}, \tilde{\psi}] ,$$

where

$$\begin{aligned} \tilde{S}_b[\tilde{A}] &:= (\text{Tr}_{\mathcal{H}} \otimes \text{id}) f(D_{\tilde{A}}/\Lambda) , \\ \tilde{S}_f[\tilde{A}, \tilde{\psi}] &:= \langle (J \otimes *)\tilde{\psi}, D_{\tilde{A}}\tilde{\psi} \rangle_Q , \end{aligned}$$

and \tilde{A} is a self-adjoint element of $\Omega_D^1 \otimes_{\text{alg}} Q$, $\tilde{\psi} \in \mathcal{H}_+ \otimes Q$, $D_{\tilde{A}} := D \otimes 1 + \tilde{A} + \epsilon'(J \otimes *)\tilde{A}(J \otimes *)^{-1}$.

Here $f(D_{\tilde{A}}/\Lambda)$ is defined in the following way: if $L^2(Q)$ is the GNS representation associated to the Haar state of Q , then $\tilde{A} + \epsilon'(J \otimes *)\tilde{A}(J \otimes *)^{-1}$ is a bounded self-adjoint operator on $\mathcal{H} \otimes L^2(Q)$ and $D_{\tilde{A}}$ is a (unbounded) self-adjoint operator on the Hilbert space $\mathcal{H} \otimes L^2(Q)$. The operator $f(D_{\tilde{A}}/\Lambda)$ is then defined using the continuous functional calculus.

By (co)invariance of the action functional we mean the property

$$\tilde{S}[\beta(A, \psi)] = S[A, \psi] \cdot 1_Q. \quad (4.3)$$

Notice that since A is a self-adjoint 1-form, $\tilde{A} = \hat{U}(A \otimes 1)\hat{U}^*$ is a self-adjoint element of $\Omega_D^1 \otimes_{\text{alg}} Q$ as required above so that $\tilde{S}[\beta(A, \psi)]$ is well defined. In the remaining part of the section we discuss the invariance of the action. We study separately the fermionic and the bosonic part.

Proposition 4.2. *If \hat{U} satisfies (2.3a) and (2.3b), then*

$$\tilde{S}_f[\beta(A, \psi)] = S_f[A, \psi] \cdot 1_Q$$

for all $(A, \psi) \in \Omega_D^{1, s.a.} \oplus \mathcal{H}_+$.

Proof. This is a simple algebraic identity. Since \hat{U} commutes with D and $J \otimes *$, we have

$$D_{\hat{U}(A \otimes 1)\hat{U}^*} = D \otimes 1 + \hat{U}(A \otimes 1)\hat{U}^* + \epsilon'(J \otimes *)\hat{U}(A \otimes 1)\hat{U}^*(J \otimes *)^{-1} = \hat{U}(D_A \otimes 1)\hat{U}^*. \quad (4.4)$$

Thus,

$$\begin{aligned} \tilde{S}_f[\beta(A, \psi)] &= \langle (J \otimes *)\hat{U}(\psi \otimes 1_Q), D_{\hat{U}(A \otimes 1)\hat{U}^*}\hat{U}(\psi \otimes 1_Q) \rangle_Q \\ &= \langle \hat{U}(J\psi \otimes 1_Q), \hat{U}(D_A\psi \otimes 1_Q) \rangle_Q \\ &= \langle J\psi, D_A\psi \rangle \cdot 1_Q = S_f[A, \psi] \cdot 1_Q, \end{aligned}$$

by the unitarity of \hat{U} . □

For the rest of the subsection, we will assume that $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ is the product of two real spectral triples, one of them being even and finite-dimensional. In fact, we will use the notations in Subsection 4.1. Moreover we assume that $\hat{U} := 1 \otimes U$, where U is a unitary corepresentation of the compact quantum group Q such that (Q, U) coacts by orientation and real structure preserving isometries on the finite dimensional spectral triple $(\mathcal{A}_2, \mathcal{H}_2, D_2, \gamma_2, J_2)$. Under these assumptions, we now establish the invariance for the bosonic part.

Lemma 4.3. *For any trace-class operator L on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$*

$$(\text{Tr}_{\mathcal{H}} \otimes \text{id})\hat{U}(L \otimes 1)\hat{U}^* = \text{Tr}_{\mathcal{H}}(L) \cdot 1_Q.$$

Proof. Let $L = L_1 \otimes L_2$ with $L_1 \in \mathcal{L}^1(\mathcal{H}_1)$ and $L_2 \in \mathcal{B}(\mathcal{H}_2)$. Since

$$\hat{U}(L \otimes 1)\hat{U}^* = L_1 \otimes U(L_2 \otimes 1)U^*,$$

by Lemma 2.6, we have:

$$\begin{aligned} (\text{Tr}_{\mathcal{H}_1 \otimes \mathcal{H}_2} \otimes \text{id})\hat{U}(L \otimes 1)\hat{U}^* &= \text{Tr}_{\mathcal{H}_1}(L_1) \cdot (\text{Tr}_{\mathcal{H}_2} \otimes \text{id})U(L_2 \otimes 1)U^* \cdot 1_Q \\ &= \text{Tr}_{\mathcal{H}_1 \otimes \mathcal{H}_2}(L) \cdot 1_Q. \end{aligned}$$

Since \mathcal{H}_2 is finite dimensional, any element of $\mathcal{L}^1(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is a finite sum of elements of the form $L := L_1 \otimes L_2$, with $L_1 \in \mathcal{L}^1(\mathcal{H}_1)$ and $L_2 \in \mathcal{B}(\mathcal{H}_2)$, and thus by the linearity of the trace, the proof is finished. □

Proposition 4.4. *For any $A \in \Omega_D^{1,s,a}$, $\tilde{S}_b[\text{Ad}_{\hat{U}}(A)] = S_b[A] \cdot 1_Q$.*

Proof. From (4.4) we have

$$\begin{aligned}\tilde{S}_b[\hat{U}(A \otimes 1)\hat{U}^*] &= (\text{Tr}_{\mathcal{H}} \otimes \text{id}) f(D_{\hat{U}(A \otimes 1)\hat{U}^*}/\Lambda) \\ &= (\text{Tr}_{\mathcal{H}} \otimes \text{id}) f(\hat{U}(D_A \otimes 1)\hat{U}^*/\Lambda) .\end{aligned}$$

By continuous functional calculus,

$$f(\hat{U}(D_A \otimes 1)\hat{U}^*/\Lambda) = \hat{U}f((D_A \otimes 1)/\Lambda)\hat{U}^* = \hat{U}(f(D_A/\Lambda) \otimes 1)\hat{U}^*$$

and applying Lemma 4.3 to the trace-class operator $L := f(D_A/\Lambda)$ we get

$$\begin{aligned}\tilde{S}_b[\hat{U}(A \otimes 1)\hat{U}^*] &= (\text{Tr}_{\mathcal{H}} \otimes \text{id}) \hat{U}(L \otimes 1)\hat{U}^* \\ &= \text{Tr}_{\mathcal{H}}(L) \cdot 1_Q \equiv \text{Tr}_{\mathcal{H}}f(D_A/\Lambda) \cdot 1_Q \\ &= S_b[A] \cdot 1_Q ,\end{aligned}$$

which concludes the proof. \square

Proposition 4.5. *The bosonic and the fermionic part of the spectral action of the Standard Model are preserved by the compact quantum group $\widetilde{\text{QISO}}^+(B_F, H_F, D_F, \gamma_F, J_F)$.*

Proof. The compact quantum group $Q := \widetilde{\text{QISO}}^+(B_F, H_F, D_F, \gamma_F, J_F)$ has a corepresentation preserving \mathcal{H}_+ and it satisfies the hypothesis of Lemma 4.1 and Prop. 4.2 and 4.4, hence the result follows. \square

5 Some remarks on real $*$ -algebras and their symmetries

In Sec. 3.2 we computed the quantum isometry group of the finite part of the Standard Model by replacing the real C^* -algebra A_F with the complex C^* -algebra B_F . Here we explain what happens if we work with A_F .

Any real $*$ -algebra \mathcal{A} (i.e. unital, associative, involutive algebra over \mathbb{R}) can be thought of as the fixed point subalgebra of its complexification $\mathcal{A}_{\mathbb{C}} = \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C}$ with respect to the involutive (conjugate-linear) real $*$ -algebra automorphism σ defined by

$$\sigma(a \otimes_{\mathbb{R}} z) = a \otimes_{\mathbb{R}} \bar{z} \quad \forall a \in \mathcal{A}, z \in \mathbb{C} , \quad (5.1)$$

that is

$$\mathcal{A} = \{a \in \mathcal{A}_{\mathbb{C}} : \sigma(a) = a\} .$$

A crucial observation is that we can characterize the automorphisms of \mathcal{A} as those automorphisms of $\mathcal{A}_{\mathbb{C}}$ which commute with σ , as proved in the following lemma.

Lemma 5.1. *For any real $*$ -algebra \mathcal{A} ,*

$$\text{Aut}(\mathcal{A}) \simeq \{\phi \in \text{Aut}(\mathcal{A}_{\mathbb{C}}) : \sigma\phi = \phi\sigma\} . \quad (5.2)$$

Proof. If φ is any (real) $*$ -algebra morphism of \mathcal{A} , $\phi(a \otimes_{\mathbb{R}} z) := \varphi(a) \otimes_{\mathbb{R}} z$ defines a (complex) $*$ -algebra morphism of $\mathcal{A}_{\mathbb{C}}$ clearly satisfying $\sigma\phi = \phi\sigma$. The map $\varphi \mapsto \phi$ gives an inclusion of the left hand side of (5.2) into the right hand side. Conversely, if $\phi \in \text{Aut}(\mathcal{A}_{\mathbb{C}})$ satisfies $\sigma\phi = \phi\sigma$, then it maps the real subalgebra $\mathcal{A} \simeq \mathcal{A} \otimes_{\mathbb{R}} 1 \subset \mathcal{A}_{\mathbb{C}}$ into itself, since

$$\sigma\phi(a \otimes_{\mathbb{R}} 1) = \phi\sigma(a \otimes_{\mathbb{R}} 1) = \phi(a \otimes_{\mathbb{R}} 1)$$

for any $a \in \mathcal{A}$. Therefore, we can define an element $\varphi \in \text{Aut}(\mathcal{A})$ by $\varphi(a) \otimes_{\mathbb{R}} 1 := \phi(a \otimes_{\mathbb{R}} 1)$.

The two group homomorphisms $\varphi \mapsto \phi$ and $\phi \mapsto \varphi$ are the inverses of each other and thus, we have the isomorphism in (5.2). \square

From a dual point of view, if $G = \text{Aut}(\mathcal{A})$, the right coaction of $C(G)$ on $\mathcal{A}_{\mathbb{C}}$ is the map $\alpha : \mathcal{A}_{\mathbb{C}} \rightarrow \mathcal{A}_{\mathbb{C}} \otimes C(G) \simeq C(G; \mathcal{A}_{\mathbb{C}})$ defined by

$$(\text{id} \otimes \text{ev}_{\phi})\alpha(a) := \phi(a), \quad \phi \in G, \quad a \in \mathcal{A}_{\mathbb{C}}.$$

We can rephrase Lemma 5.1 as follows.

Lemma 5.2. *For a finite dimensional real C^* -algebra \mathcal{A} , the condition $\sigma\phi = \phi\sigma \forall \phi \in G$ is equivalent to*

$$(\sigma \otimes *_{C(G)})\alpha = \alpha\sigma.$$

Proof. Let $\alpha_{\phi}\sigma = (\sigma \otimes \text{ev}_{\phi} *_{C(G)})\alpha$ and $\phi \in G$, $a \in \mathcal{A}_{\mathbb{C}}$. Let us suppose that $(\sigma \otimes *_{C(G)})\alpha = \alpha\sigma$. Then $\sigma\phi(a) = (\text{id} \otimes \text{ev}_{\phi})\alpha\sigma(a) = (\sigma \otimes \text{ev}_{\phi} *_{C(G)})\alpha(a) = (\sigma \otimes *_{\mathbb{C}} \text{ev}_{\phi})\alpha(a) = \phi\sigma(a)$ by the antilinearity of σ . Conversely, if $\sigma\phi = \phi\sigma \forall \phi \in G$ then for all ϕ , $(\text{id} \otimes \text{ev}_{\phi})\alpha(\sigma(a)) = (\sigma \otimes \text{ev}_{\phi})(\alpha(a))$. Thus, $(\sigma \otimes \text{ev}_{\phi} *_{C(G)})\alpha(a) = (\sigma \otimes *_{\mathbb{C}} \text{ev}_{\phi})\alpha(a) = \sigma((\text{id} \otimes \text{ev}_{\phi})\alpha(a)) = \sigma\phi(a) = \phi\sigma(a) = (\text{id} \otimes \text{ev}_{\phi})\alpha(\sigma(a))$. As $\{\text{ev}_{\phi} : \phi \in G\}$ separates points on G , this proves $(\sigma \otimes *_{C(G)})\alpha = \alpha\sigma$. \square

Motivated by this lemma, we consider the category $\mathfrak{C}_{J, \mathbb{R}}$ of CQGs coacting by orientation and real structure preserving isometries via a unitary corepresentation U (in the sense of Def. 2.12) on the spectral triple $(B_F, H_F, D_F, \gamma_F, J_F)$ whose adjoint coaction Ad_U can be extended to a coaction α on $(A_F)_{\mathbb{C}} = A_F \otimes_{\mathbb{R}} \mathbb{C}$ satisfying

$$(\sigma \otimes *)\alpha = \alpha\sigma. \quad (5.3)$$

We notice that it is a subcategory of \mathfrak{C}_J : objects of $\mathfrak{C}_{J, \mathbb{R}}$ are those objects of \mathfrak{C}_J compatible with σ in the sense explained above, and the morphisms in the two categories are the same.

Thus any object, say Q , of $\mathfrak{C}_{J, \mathbb{R}}$ satisfies the relations of the universal object $\widehat{\text{QISO}}_J^+(D_F)$ of \mathfrak{C}_J in Prop. 3.4. In the rest of this subsection, with a slight abuse of notation, we will continue to denote the generators of Q by the same symbols as in Prop. 3.4.

Theorem 5.3. *A compact quantum group Q is an object in $\mathfrak{C}_{J, \mathbb{R}}$ if and only if the generators satisfy*

$$(T_m)_{jk}(T_m)_{j'k'}^*(T_m)_{j''k''} = (T_m)_{j''k''}(T_m)_{j'k'}^*(T_m)_{jk} \quad (5.4)$$

for all $m = 1, \dots, n$ and all $j, j', j'', k, k', k'' \in \{1, 2, 3\}$.

Proof. The real algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ is the fixed point subalgebra of $(A_F)_{\mathbb{C}} \simeq \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C})$ with respect to the automorphism

$$\sigma(\lambda, \lambda', q, m, m') = (\bar{\lambda}', \bar{\lambda}, \sigma_2 \bar{q} \sigma_2, \bar{m}', \bar{m}),$$

where σ_2 is the second Pauli matrix:

$$\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

It is easy to check that $q \in M_2(\mathbb{C})$ satisfies $\sigma_2 \bar{q} \sigma_2 = q$ if and only if it is of the form (3.2), and that under the isomorphism (3.7) \mathbb{C} is identified with the real subalgebra of $\mathbb{C} \oplus \mathbb{C}$ with elements $(\lambda, \bar{\lambda})$ and $M_3(\mathbb{C})$ with the real subalgebra of $M_3(\mathbb{C}) \oplus M_3(\mathbb{C})$ with elements (m, \bar{m}) .

The coaction on the factor $B_F \subset (A_F)_{\mathbb{C}}$ is given by (3.14), and an extension $\widetilde{\text{Ad}}_U$ to $(A_F)_{\mathbb{C}}$ satisfying (5.3) exists if and only if

$$\begin{aligned} \widetilde{\text{Ad}}_U(0, 0, 0, 0, e_{ij}) &= (\sigma \otimes *) \widetilde{\text{Ad}}_U \sigma(0, 0, 0, 0, e_{ij}) \\ &= (\sigma \otimes *) \widetilde{\text{Ad}}_U(0, 0, 0, \bar{e}_{ij}, 0) \\ &= (\sigma \otimes *) (\text{Ad}_U(\langle 0, 0, 0, e_{ij} \rangle), 0) \\ &= (\sigma \otimes *) \sum_{k,l=1,2,3} (0, 0, 0, e_{kl}, 0) \otimes (T_1)_{ki}^* (T_1)_{lj} \\ &= \sum_{k,l=1,2,3} (0, 0, 0, 0, e_{kl}) \otimes (T_1)_{lj}^* (T_1)_{ki}. \end{aligned}$$

The only conditions left to impose is that this extension is a coaction of a CQG. As it is already a coaction on B_F , we need to impose it for the coaction on the second copy of $M_3(\mathbb{C})$, which has to be preserved by $\widetilde{\text{Ad}}_U$. At this point, we note that as $\widetilde{\text{Ad}}_U$ is an extension of Ad_U , which preserves the trace on the first copy of $M_3(\mathbb{C})$, the formula $\widetilde{\text{Ad}}_U(0, 0, 0, 0, e_{ij}) = \sum_{k,l=1,2,3} (0, 0, 0, 0, e_{kl}) \otimes (T_1)_{lj}^* (T_1)_{ki}$ forces $\widetilde{\text{Ad}}_U$ to preserve the trace on the second copy of $M_3(\mathbb{C})$. Thus, by Theorem 4.1 of [37], it suffices to impose the conditions (4.1-4.5) in that paper with a_{ij}^{kl} replaced by $(T_m)_{lj}^* (T_m)_{ki}$. It is easy to check that (4.3-4.5) are automatically satisfied. The only non trivial conditions come from (4.1) and (4.2).

From (4.1), we get

$$\sum_{v=1}^3 (T_m)_{vj}^* (T_m)_{ki} (T_m)_{ls}^* (T_m)_{vr} = \delta_{jr} (T_m)_{ls}^* (T_m)_{ki} \quad (5.5)$$

From (4.2), we get the same relation with $(T_m)^t$ instead of T_m . Now we show that (5.5) and (5.4) are equivalent, which will finish the proof since if T_m satisfies (5.4), then $(T_m)^t$ satisfies it too.

If we multiply both sides of (5.5) by $(T_m)_{qj}$ from the left and sum over j , we get

$$\sum_{v=1}^3 \delta_{vq} (T_m)_{ki} (T_m)_{ls}^* (T_m)_{vr} = \sum_{j=1}^3 \delta_{jr} (T_m)_{qj} (T_m)_{ls}^* (T_m)_{ki}$$

using biunitarity of T_m . The last equation is clearly equivalent to (5.4). To prove that (5.4) implies (5.5), it is enough to multiply both sides by $(T_m)_{j''k''}$ from the left, then sum over j'' and use the biunitarity of T_m again. \square

It is easy to check that (5.4) defines a Woronowicz C^* -ideal, and hence the quotient of $\widetilde{\text{QISO}}_J^+(D_F)$ by (5.4) is a CQG. This leads to the following corollary.

Corollary 5.4. *Let $\widetilde{\text{QISO}}_{\mathbb{R}}^+(D_F)$ be the quantum subgroup of the CQG $\widetilde{\text{QISO}}_J^+(D_F)$ in Prop. 3.4 defined by the relations (5.4). Then $\widetilde{\text{QISO}}_{\mathbb{R}}^+(D_F)$ is the universal object in the category $\mathfrak{C}_{J,\mathbb{R}}$.*

Motivated by (5.4), we give the following definition.

Definition 5.5. For a fixed N , we call $A_u^*(N)$ the universal unital C^* -algebra generated by a $N \times N$ biunitary $u = ((u_{ij}))$ with relations

$$ab^*c = cb^*a, \quad \forall a, b, c \in \{u_{ij}, i, j = 1, \dots, N\}. \quad (5.6)$$

$A_u^*(N)$ is a CQG with coproduct given by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$.

We will call $A_u^*(N)$ the N -dimensional *half-liberated unitary group*. This is similar to the half-liberated orthogonal group $A_o^*(N)$, that can be obtained by imposing the further relation $a = a^*$ for all $a \in \{u_{ij}, i, j = 1, \dots, N\}$ (cf. [5]).

Remark 5.6. We notice that there are two other possible ways to “half-liberate” the free unitary group. Instead of $ab^*c = cb^*a$ (which by adjunction is equivalent to $a^*bc^* = c^*ba^*$), one can consider respectively the relation $a^*bc = cba^*$ (which is equivalent to $abc^* = c^*ba$ and to the adjoints $ab^*c^* = c^*b^*a$ and $a^*b^*c = cb^*a^*$) or $abc = cba$ (equivalent to $a^*b^*c^* = c^*b^*a^*$) for any triple $a, b, c \in \{u_{ij}, i, j = 1, \dots, N\}$.

Like $A_o^*(N)$, the projective version of $A_u^*(N)$ is also commutative, as proved in the next proposition.

Proposition 5.7. The CQG $PA_u^*(N)$ is isomorphic to $C(PU(N))$.

Proof. We recall (Rem. 2.8) that for a CQG Q generated by a biunitary $u = ((u_{ij}))$, the projective version is the C^* -subalgebra generated by products $u_{ij}^* u_{kl}$.

Clearly $C(U(N))$ is a quantum subgroup of $A_u^*(N)$, and the latter is a quantum subgroup of $A_u(N)$. Thus, $C(PU(N))$ is a quantum subgroup of $PA_u^*(N)$, which is a quantum subgroup of $PA_u(N)$. Since the abelianization of $PA_u(N)$ is exactly $C(PU(N))$, any commutative (as a C^* -algebra) quantum subgroup of $PA_u(N)$ containing $C(PU(N))$ coincides with $C(PU(N))$. Thus, the proof will be over if we can show that the C^* -algebra of $PA_u(N)$ is commutative, i.e. $PA_u(N)$ is the space of continuous functions on a compact group. This is a simple computation. Using first (5.6) and then its adjoint we get:

$$\begin{aligned} (u_{ij}^* u_{kl})(u_{pq}^* u_{rs}) &= u_{ij}^*(u_{kl} u_{pq}^* u_{rs}) = u_{ij}^*(u_{rs} u_{pq}^* u_{kl}) \\ &= (u_{ij}^* u_{rs} u_{pq}^*) u_{kl} = (u_{pq}^* u_{rs} u_{ij}^*) u_{kl} \\ &= (u_{pq}^* u_{rs})(u_{ij}^* u_{kl}). \end{aligned}$$

This proves that the generators of $PA_u(N)$ commute, which concludes the proof. \square

In complete analogy with (3.12), if we call $Q_n^*(n')$ the amalgamated free product of n copies of $A_u^*(n')$ over the common Woronowicz C^* -subalgebra $C(PU(n'))$, then we have:

Corollary 5.8. $\widetilde{\text{QISO}}_{\mathbb{R}}^+(D_F)$ is a quantum subgroup of the free product

$$\underbrace{C(U(1)) * C(U(1)) * \dots * C(U(1))}_{n+1} * Q_n^*(3) * A_u(n)$$

The Woronowicz C^* -ideal of this CQG defining $\widetilde{\text{QISO}}_{\mathbb{R}}^+(D_F)$ is determined by (3.10a) and (3.10b).

As in the complex case, let us denote by $\text{QISO}_{\mathbb{R}}^+(D_F)$ the C^* -subalgebra of $\widetilde{\text{QISO}}_{\mathbb{R}}^+(D_F)$ generated by $\langle \xi \otimes 1, \text{Ad}_{U_{\mathbb{R}}}(a)(\eta \otimes 1) \rangle$, where $a \in B_F$, $\xi, \eta \in H_F$ and $U_{\mathbb{R}}$ is the corepresentation of $\widetilde{\text{QISO}}_{\mathbb{R}}^+(D_F)$. An immediate corollary of Prop. 5.7 and Corollary 5.8 is the following.

Corollary 5.9. $\text{QISO}_{\mathbb{R}}^+(D_F) = C(U(1)) * C(PU(3))$.

Remark 5.10. Since $\widetilde{\text{QISO}}_{\mathbb{R}}^+(D_F)$ is a quantum subgroup of $\widetilde{\text{QISO}}_J^+(D_F)$, its coaction still preserves the spectral action.

A detailed study of quantum automorphisms for finite-dimensional real C^* -algebras, along the lines of the discussion in this section, will be reported elsewhere.

6 Proof of Proposition 3.4

In this section, we prove the main result, that is, Proposition 3.4. Throughout this section, (Q, U) will denote an object in \mathfrak{C}_J . We start by exploiting the conditions regarding γ_F and J_F , then we use the conditions regarding D_F and Ad_U to get a neater expression for U in Lemma 6.2, 6.3 and 6.4 and then using these simplified expressions in the next Lemmas, we derive the desired form of U from which we can identify the quantum isometry group. We will use Remark 3.1 in this section without mentioning it. Recall that $\mathcal{B}(H_F) = M_2(\mathbb{C}) \otimes M_4(\mathbb{C}) \otimes M_4(\mathbb{C}) \otimes M_n(\mathbb{C})$, where n is the number of generations.

Lemma 6.1. $U \in \mathcal{B}(H_F) \otimes Q$ satisfies $(\gamma_F \otimes 1)U = U(\gamma_F \otimes 1)$ and $(J_0 \otimes 1)\overline{U} = U(J_0 \otimes 1)$ iff

$$U = \sum_{IJ} (e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes e_{i_3 j_3} \otimes e_{i_4 j_4}) \otimes u_{IJ} + \sum_{IJ} (e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes e_{i_3+2, j_3+2} \otimes e_{i_4 j_4}) \otimes \overline{u}_{IJ}, \quad (6.1)$$

where the multi-indices $I = (i_1, \dots, i_4)$, $J = (j_1, \dots, j_4)$, etc. run in $\{1, 2\} \times \{1, 2, 3, 4\} \times \{1, 2\} \times \{1, 2, \dots, n\}$.

Proof. The condition $(\gamma_F \otimes 1)U = U(\gamma_F \otimes 1)$ implies that $u_{i_1, j_1, i_2, j_2, i_3, j_3, i_4, j_4} = 0$ unless i_3, j_3 are both greater or equal than 2 or both less or equal than 3. Using the reduced form of U obtained from this observation, we impose $(J_0 \otimes 1)\overline{U} = U(J_0 \otimes 1)$ and get $u_{i_1, j_1, i_2, j_2, i_3, j_3, i_4, j_4} = (u_{i_1, j_1, i_2, j_2, i_3-2, j_3-2, i_4, j_4})^*$ for all $i_3, j_3 \geq 3$, which proves the Lemma. \square

Let V_1, V_2, V_3, V_4 denote the subspaces $(e_{11} \otimes e_{11} \otimes 1 \otimes 1)\mathcal{H}$, $(e_{22} \otimes e_{11} \otimes 1 \otimes 1)\mathcal{H}$, $(e_{11} \otimes (1 - e_{11}) \otimes 1 \otimes 1)\mathcal{H}$, and $(e_{22} \otimes (1 - e_{11}) \otimes 1 \otimes 1)\mathcal{H}$ respectively.

Lemma 6.2. If U is of the form (6.1) and commutes with D_F , the subspaces V_i , $i = 1, 2, 3, 4$ are kept invariant by U and thus (6.1) becomes

$$U = \sum_{i=1,2} e_{ii} \otimes e_{11} \otimes \begin{pmatrix} \alpha_{11}^i & \alpha_{12}^i & 0 & 0 \\ \alpha_{21}^i & \alpha_{22}^i & 0 & 0 \\ 0 & 0 & \overline{\alpha}_{11}^i & \overline{\alpha}_{12}^i \\ 0 & 0 & \overline{\alpha}_{21}^i & \overline{\alpha}_{22}^i \end{pmatrix} + \sum_{\substack{i=1,2 \\ j,k=1,2,3}} e_{ii} \otimes e_{j+1, k+1} \otimes \begin{pmatrix} \beta_{11}^{i,j,k} & \beta_{12}^{i,j,k} & 0 & 0 \\ \beta_{21}^{i,j,k} & \beta_{22}^{i,j,k} & 0 & 0 \\ 0 & 0 & \overline{\beta}_{11}^{i,j,k} & \overline{\beta}_{12}^{i,j,k} \\ 0 & 0 & \overline{\beta}_{21}^{i,j,k} & \overline{\beta}_{22}^{i,j,k} \end{pmatrix} \quad (6.2)$$

where, as in (3.5) we identify $M_4(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes Q$ with $M_{4n}(Q)$, we call $\alpha_{j_1 k_1}^i$ is the $n \times n$ matrix with entries $(\alpha_{j_1 k_1}^i)_{j_2 k_2} := u_{JK}$ with $J = (i, 1, j_1, j_2)$ and $K = (i, 1, k_1, k_2)$ and we call $\beta_{j_1 k_1}^{i, j_0, k_0}$ the $n \times n$ matrix with entries $(\beta_{j_1 k_1}^{i, j_0, k_0})_{j_2 k_2} := u_{JK}$ with $J = (i, j_0 + 1, j_1, j_2)$ and $K = (i, k_0 + 1, k_1, k_2)$.

Proof. The subspaces $V_i, i = 1, 2, 3, 4$ are D_F -invariant and correspond to distinct sets of eigenvalues (masses of the generations of ν , e , u and d respectively). Since $(D_F \otimes 1)U = U(D_F \otimes 1)$ these four subspaces must be preserved by U and this completes the proof of the lemma. \square

Lemma 6.3. *Let Q be any CQG with U as in (6.1) and satisfying (3.9b). Then each one of the four summands in $B_F = \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ is a coinvariant subalgebra under the adjoint coaction $\text{Ad}_U(a) = U(a \otimes 1)U^*$ of Q .*

Proof. We start with the basis element $\langle 0, 1, 0, 0 \rangle$ of the second copy of \mathbb{C} . Equation (3.9b) means that

$$\begin{aligned} \text{Ad}_U(\langle 0, 1, 0, 0 \rangle) &= \langle 1, 0, 0, 0 \rangle \otimes a^{\langle 1, 0, 0, 0 \rangle} + \langle 0, 1, 0, 0 \rangle \otimes a^{\langle 0, 1, 0, 0 \rangle} \\ &\quad + \sum_{i,j=1,2} \langle 0, 0, e_{ij}, 0 \rangle \otimes a^{\langle 0, 0, e_{ij}, 0 \rangle} + \sum_{i,j=1,2,3} \langle 0, 0, 0, e_{ij} \rangle \otimes a^{\langle 0, 0, 0, e_{ij} \rangle}, \end{aligned} \quad (6.3)$$

where $a^{\langle \cdot \rangle}$ are some elements of Q .

By (6.2), $U(\langle 0, 1, 0, 0 \rangle \otimes 1)U^*$ has e_{22} in the first position and e_{jk} in the third, with $j, k = 3, 4$. Therefore, $U(\langle 0, 1, 0, 0 \rangle \otimes 1)U^*$ vanishes on the subspaces $(e_{11} \otimes 1 \otimes e_{44} \otimes 1)H_F$, $(1 \otimes 1 \otimes e_{11} \otimes 1)H_F$ and $(1 \otimes (1 - e_{11}) \otimes (e_{22} + e_{33}) \otimes 1)H_F$. Applying (6.3) on these three subspaces and using (3.8) we get respectively:

$$\begin{aligned} 0 &= (e_{11} \otimes 1 \otimes e_{44} \otimes 1) \otimes a^{\langle 1, 0, 0, 0 \rangle} + 0 + 0 + 0, \\ 0 &= 0 + 0 + \sum_{i,j=1,2} \langle 0, 0, e_{ij}, 0 \rangle \otimes a^{\langle 0, 0, e_{ij}, 0 \rangle} + 0, \\ 0 &= 0 + 0 + 0 + \sum_{i,j=1,2,3} \langle 0, 0, 0, e_{ij} \rangle \otimes a^{\langle 0, 0, 0, e_{ij} \rangle}. \end{aligned}$$

Therefore $a^{\langle 1, 0, 0, 0 \rangle} = a^{\langle 0, 0, e_{ij}, 0 \rangle} = a^{\langle 0, 0, 0, e_{ij} \rangle} = 0$ and $\text{Ad}_U(\langle 0, 1, 0, 0 \rangle) \subset \langle 0, 1, 0, 0 \rangle \otimes Q$. The proof for the other three factors is similar.

For the rest of the proof, let $\lambda \in \mathbb{C}$, $q \in M_2(\mathbb{C})$, $m \in M_3(\mathbb{C})$ be arbitrary.

$U(\langle 1, 0, 0, 0 \rangle \otimes 1)U^*$ vanishes on the subspaces $(e_{22} \otimes (1 - e_{11}) \otimes e_{44} \otimes 1)H_F$, $(1 \otimes e_{11} \otimes e_{11} \otimes 1)H_F$ and $(1 \otimes (1 - e_{11}) \otimes e_{22} \otimes 1)H_F$ and hence this implies respectively that the coefficients of $\langle 0, \lambda, 0, 0 \rangle$, $\langle 0, 0, q, 0 \rangle$, $\langle 0, 0, 0, m \rangle$ in $\text{Ad}_U(\langle 1, 0, 0, 0 \rangle)$ are zero.

$U(\langle 0, 0, q, 0 \rangle \otimes 1)U^*$ vanishes on the subspaces $(e_{11} \otimes 1 \otimes e_{44} \otimes 1)H_F$, $(e_{22} \otimes 1 \otimes e_{44} \otimes 1)H_F$ and $(1 \otimes (1 - e_{11}) \otimes e_{33} \otimes 1)H_F$ and hence this implies respectively that the coefficients of $\langle \lambda, 0, 0, 0 \rangle$, $\langle 0, \lambda, 0, 0 \rangle$, $\langle 0, 0, 0, m \rangle$ in $\text{Ad}_U(\langle 0, 0, q, 0 \rangle)$ are zero.

Finally, $U(\langle 0, 0, 0, m \rangle \otimes 1)U^*$ vanishes on the subspaces $(e_{11} \otimes e_{11} \otimes e_{44} \otimes 1)H_F$, $(e_{22} \otimes e_{11} \otimes e_{44} \otimes 1)H_F$ and $(1 \otimes e_{11} \otimes e_{11} \otimes 1)H_F$ which implies respectively that the coefficients of $\langle \lambda, 0, 0, 0 \rangle$, $\langle 0, \lambda, 0, 0 \rangle$, $\langle 0, 0, q, 0 \rangle$ in $\text{Ad}_U(\langle 0, 0, 0, m \rangle)$ are zero. \square

Lemma 6.4. *If (3.9b) is satisfied, the matrices $\alpha_{j_1 k_1}^i$ and $\beta_{j_1 k_1}^{i, j_0, k_0}$ in (6.2) are zero for all $j_1 \neq k_1$.*

Proof. We use Lemma 6.3. Since $\text{Ad}_U(\langle 1, 0, 0, 0 \rangle) \subset \langle 1, 0, 0, 0 \rangle \otimes Q$, it is easy to see that $(e_{ii} \otimes e_{11} \otimes e_{11} \otimes 1 \otimes 1_Q) \text{Ad}_U(\langle 1, 0, 0, 0 \rangle)$ equals zero for all $i = 1, 2$. On the other hand, straightforward computation gives

$$(e_{ii} \otimes e_{11} \otimes e_{11} \otimes 1 \otimes 1_Q) \text{Ad}_U(\langle 1, 0, 0, 0 \rangle) = e_{ii} \otimes e_{11} \otimes e_{11} \otimes \alpha_{12}^i (\alpha_{12}^i)^* = 0,$$

from which it follows that $\alpha_{12}^i = 0$ for all $i = 1, 2$. Similarly,

$$(1 \otimes e_{11} \otimes e_{22} \otimes 1 \otimes 1_Q) \text{Ad}_U(\langle 0, 0, e_{ii}, 0 \rangle) = e_{ii} \otimes e_{11} \otimes e_{22} \otimes \alpha_{21}^i (\alpha_{21}^i)^* = 0$$

gives $\alpha_{21}^i = 0$ for all $i = 1, 2$. Finally, $\text{Ad}_U(\langle 0, 0, 0, e_{k_0 l_0} \rangle)$ applied to the projections $1 \otimes 1 \otimes e_{11} \otimes 1$ and $1 \otimes 1 \otimes e_{44} \otimes 1$, we get the conditions

$$\beta_{12}^{i,j_0,k_0} (\beta_{12}^{i,l_0,n_0})^* = \overline{\beta_{21}^{i,j_0,k_0}} (\beta_{21}^{i,l_0,n_0})^t = 0$$

for all i, j_0, k_0, l_0, n_0 . In particular setting $j_0 = l_0$ and $k_0 = n_0$ we get $\beta_{12}^{i,j_0,k_0} = \beta_{21}^{i,j_0,k_0} = 0$. \square

Now we impose $U(D_F \otimes 1) = (D_F \otimes 1)U$, with D_F as in (3.5), U as in (6.2) and using Lemma 6.4.

Lemma 6.5. *Any U of the form (6.2), and with $\alpha_{j_1 k_1}^i = \beta_{j_1 k_1}^{i,j_0,k_0} = 0$ for all $j_1 \neq k_1$, satisfies $U(D_F \otimes 1) = (D_F \otimes 1)U$ if and only if*

1. all α_{ss}^2 and $\beta_{rr}^{1,j,k}$ are diagonal $n \times n$ matrices,
2. $\alpha_{22}^2 = \overline{\alpha_{11}^2}$, $\beta_{22}^{1,j,k} = \overline{\beta_{11}^{1,j,k}}$, $\beta_{22}^{2,j,r} = \overline{\beta_{11}^{2,j,r}}$,
3. $\alpha_{11}^1 \Upsilon_\nu = \Upsilon_\nu \alpha_{11}^1 = \overline{\alpha_{22}^1} \Upsilon_\nu = \Upsilon_\nu \overline{\alpha_{22}^1}$, $\alpha_{22}^1 \Upsilon_R = \Upsilon_R \overline{\alpha_{22}^1}$
4. $C^* \beta_{11}^{2,j,k} C$ is a diagonal matrix.

Proof. The condition $U(D_F \otimes 1) = (D_F \otimes 1)U$ is equivalent to the following sets of equations:

$$\alpha_{11}^1 \Upsilon_\nu = \Upsilon_\nu \overline{\alpha_{22}^1}, \quad \overline{\alpha_{22}^1} \Upsilon_\nu = \Upsilon_\nu \alpha_{11}^1, \quad \alpha_{22}^1 \Upsilon_R = \Upsilon_R \overline{\alpha_{22}^1}, \quad (6.4a)$$

$$\alpha_{11}^2 \Upsilon_e = \Upsilon_e \overline{\alpha_{22}^2}, \quad \alpha_{22}^2 \Upsilon_e = \Upsilon_e \overline{\alpha_{11}^2}, \quad \beta_{11}^{1,j,k} \Upsilon_u = \Upsilon_u \overline{\beta_{22}^{1,j,k}}, \quad (6.4b)$$

$$\overline{\beta_{22}^{1,j,k}} \Upsilon_u = \Upsilon_u \beta_{11}^{1,j,k}, \quad \beta_{11}^{2,j,k} \Upsilon_d = \Upsilon_d \overline{\beta_{22}^{2,j,k}}, \quad \overline{\beta_{22}^{2,j,k}} \Upsilon_d = \Upsilon_d \beta_{11}^{2,j,k}, \quad (6.4c)$$

Actually, there are additional 9 relations that — recalling that Υ_x ($x = e, u, d, \nu$) are positive, Υ_e, Υ_u are diagonal and Υ_R is symmetric — turn out to be the “bar” of previous ones and hence they do not give any new information.

From the first two equations in (6.4a), we deduce that α_{11}^1 commute with Υ_ν^2 :

$$\alpha_{11}^1 \Upsilon_\nu^2 = \Upsilon_\nu \overline{\alpha_{22}^1} \Upsilon_\nu = \Upsilon_\nu^2 \alpha_{11}^1,$$

and hence it commutes with its positive square root Υ_ν . Similarly α_{11}^1 commutes with Υ_ν and the conditions (6.4a) turn out to be equivalent to point 3. of the Lemma.

In a similar way from (6.4b) and (6.4c) we deduce that all α_{ss}^2 commute with Υ_e^2 and all $\beta_{rr}^{1,j,k}$ commute with Υ_u^2 . Since Υ_x^2 ($x = e, u$) are diagonal with distinct eigenvalues, we deduce that all α_{ss}^2 and $\beta_{rr}^{1,j,k}$ must be diagonal $n \times n$ matrices. This proves 1.

As all α_{ss}^2 and $\beta_{rr}^{1,j,k}$ are diagonal, (6.4b) implies that $\alpha_{22}^2 = \overline{\alpha_{11}^2}$ and $\beta_{22}^{1,j,k} = \overline{\beta_{11}^{1,j,k}}$, where we have used that Υ_e and Υ_u are diagonal invertible matrices. Thus the first two equations of 2. are proved.

The second and third equation of (6.4c) implies respectively

$$\beta_{22}^{2,j,k} = (\Upsilon_d^t)^{-1} \overline{\beta_{11}^{2,j,k}} \Upsilon_d^t, \quad \beta_{22}^{2,j,k} = \Upsilon_d^t \overline{\beta_{11}^{2,j,k}} (\Upsilon_d^t)^{-1}. \quad (6.5)$$

These two equations taken together means

$$(\Upsilon_d^t)^{-1} \overline{\beta_{11}^{2,j,k}} \Upsilon_d^t = \Upsilon_d^t \overline{\beta_{11}^{2,j,k}} (\Upsilon_d^t)^{-1}.$$

Thus, $\beta_{11}^{2,j,k} = \Upsilon_d \Upsilon_d^* \beta_{11}^{2,j,k} (\Upsilon_d \Upsilon_d^*)^{-1}$. But, $\Upsilon_d \Upsilon_d^* = C \delta_{\downarrow} C^* C \delta_{\downarrow}^* C'^* = C \delta_{\downarrow} \delta_{\downarrow}^* C'^*$. Thus, $C^* \beta_{11}^{2,j,k} C$ commutes with $\delta_{\downarrow} \delta_{\downarrow}^*$. As the latter is a diagonal matrix with distinct eigenvalues, $C^* \beta_{11}^{2,j,k} C$ is a diagonal matrix. The fact that $C^* \beta_{11}^{2,j,k} C$ is diagonal implies that $(\Upsilon_d^t)^{-1} \overline{\beta_{11}^{2,j,k}} \Upsilon_d^t = \Upsilon_d^t \overline{\beta_{11}^{2,j,k}} (\Upsilon_d^t)^{-1}$. Thus, (6.5) is equivalent to 4.

The equation remaining to be proved is the third equation of 2. which follows by part 4. and (6.5). Indeed, by (6.5), $\beta_{22}^{2,j,r} = \Upsilon_d^t \overline{\beta_{11}^{2,j,r}} (\Upsilon_d^t)^{-1} = \overline{C} \delta_{\downarrow} (C)^t \overline{\beta_{11}^{2,j,r}} \overline{C} (\delta_{\downarrow})^{-1} (C)^t$.

As $(C)^* \beta_{11}^{2,j,r} C$ is diagonal by part 4., $(C)^t \overline{\beta_{11}^{2,j,r}} \overline{C}$ is also diagonal and hence it commutes with δ_{\downarrow} and thus we get $\beta_{22}^{2,j,r} = \overline{\beta_{11}^{2,j,r}}$.

Conversely, if 1. - 4. of this Lemma are satisfied, then it can be easily verified that (6.4a), (6.4b) and (6.4c) are satisfied and hence U commutes with D_F . \square

In view of Lemma 6.5, we define elements x_k and 3×3 matrices T_m by

$$\alpha_{11}^2 = \sum_{k=1}^n e_{kk} \otimes x_k, \quad \beta_{11}^{1,j,k} = \sum_{m=1}^n e_{mm} \otimes (T_m)_{j,k}.$$

Hence, by part 2. of Lemma 6.5,

$$\beta_{22}^{1,j,k} = \sum_{m=1}^n e_{mm} \otimes (\overline{T}_m)_{j,k}.$$

Moreover, let

$$X(s, m) = \sum e_{ij} \otimes (\beta_{11}^{2,i,j})_{s,m}.$$

Lemma 6.6. *If U is a unitary corepresentation satisfying the hypothesis of Lemma 6.5, then the matrices α_{rr}^i , T_m and $X(m, m)$ are biunitaries. In particular, $\{x_1, x_2, \dots, x_n\}$ are unitary elements.*

Proof. The condition $UU^* = 1 \otimes 1$ implies that for $r = 1, 2$,

$$\begin{aligned} \alpha_{rr}^i (\alpha_{rr}^i)^* &= \overline{\alpha_{rr}^i} (\overline{\alpha_{rr}^i})^* = 1, \\ \sum_k \beta_{rr}^{i,j,k} (\beta_{rr}^{i,l,k})^* &= \sum_k \overline{\beta_{rr}^{i,j,k}} (\overline{\beta_{rr}^{i,l,k}}) = \delta_{jl}. \end{aligned}$$

Similarly, from $U^*U = 1 \otimes 1$ we get the relations

$$\begin{aligned} (\alpha_{rr}^i)^* \alpha_{rr}^i &= (\overline{\alpha_{rr}^i})^* \overline{\alpha_{rr}^i} = 1, \\ \sum_k (\beta_{rr}^{i,l,k})^* \beta_{rr}^{i,j,k} &= \sum_k (\overline{\beta_{rr}^{i,l,k}})^* \overline{\beta_{rr}^{i,j,k}} = \delta_{jl}. \end{aligned}$$

Thus, the matrices α_{rr}^i , T_m and $X(m, m)$ are biunitaries. \square

We note that in Lemma 6.4 we provide a necessary condition for (3.9b). The next Lemma gives conditions that are necessary and sufficient.

Lemma 6.7. Assume U satisfies the hypothesis of Lemma 6.5 and 6.6. The condition (3.9b) is satisfied, i.e. the coaction Ad_U preserves the subalgebra B_F , iff there exists a unitary x_0 such that

$$\alpha_{11}^1 = \text{diag}(x_0 x_1, \dots, x_0 x_n), \quad \alpha_{22}^2 = \text{diag}(x_1^*, \dots, x_n^*), \quad (6.6a)$$

$$\beta_{11}^{2,i,j} = \text{diag}(x_0^*(T_1)_{i,j}, \dots, x_0^*(T_n)_{i,j}), \quad (6.6b)$$

$$\sum_{m=1}^n C_{rm} \overline{C}_{sm}(T_m)_{j,k} = 0 \quad \forall r \neq s \quad (r, s = 1, \dots, n; j, k = 1, 2, 3), \quad (6.6c)$$

$$(T_m^*)_{i,j}(T_m)_{k,l} = (T_{m'}^*)_{i,j}(T_{m'})_{k,l} \quad \forall m, m' \quad (i, j, k, l = 1, 2, 3, m, m' = 1, \dots, n), \quad (6.6d)$$

and the adjoint coaction is

$$\text{Ad}_U(\langle 0, 0, e_{12}, 0 \rangle) = \langle 0, 0, e_{12}, 0 \rangle \otimes x_0, \quad (6.7a)$$

$$\text{Ad}_U(\langle 0, 0, e_{21}, 0 \rangle) = \langle 0, 0, e_{21}, 0 \rangle \otimes x_0^*, \quad (6.7b)$$

$$\text{Ad}_U(\langle 0, 0, 0, e_{ij} \rangle) = \sum_{kl} \langle 0, 0, 0, e_{kl} \rangle \otimes ((T_1)_{k,i})^*(T_1)_{l,j}. \quad (6.7c)$$

Moreover, $\langle 1, 0, 0, 0 \rangle$, $\langle 0, 1, 0, 0 \rangle$ and $\langle 0, 0, e_{ii}, 0 \rangle$ ($i = 1, 2$) are coinvariant.

Proof. We use the notations of the previous lemmas. The coinvariance of $\langle 1, 0, 0, 0 \rangle$, $\langle 0, 1, 0, 0 \rangle$ and $\langle 0, 0, e_{ii}, 0 \rangle$ ($i = 1, 2$) follows automatically from unitarity of U . Since

$$\text{Ad}_U(\langle 0, 0, e_{12}, 0 \rangle) = e_{12} \otimes e_{11} \otimes e_{11} \otimes \alpha_{11}^1 (\alpha_{11}^2)^* + \sum_{ijk} e_{12} \otimes e_{i+1,k+1} \otimes e_{11} \otimes \beta_{11}^{1,i,j} (\beta_{11}^{2,k,j})^*,$$

condition (3.9b) implies that there exists $x_0 \in Q$ such that

$$\alpha_{11}^1 (\alpha_{11}^2)^* = \sum_{i=1}^n e_{ii} \otimes x_0, \quad \sum_j \beta_{11}^{1,i,j} (\beta_{11}^{2,k,j})^* = \delta_{i,k} \left(\sum_{i=1}^n e_{ii} \otimes x_0 \right). \quad (6.8)$$

Unitarity of α_{rr}^i implies unitarity of x_0 . Moreover, we have $\alpha_{11}^1 = \text{diag}(x_0 x_1, \dots, x_0 x_n)$.

Using the relation $\alpha_{11}^2 = \overline{\alpha_{22}^2}$ in Lemma 6.5, we deduce that $\alpha_{22}^2 = \sum_{k=1}^n e_{kk} \otimes x_k^*$.

We get $\text{Ad}_U(\langle 0, 0, e_{12}, 0 \rangle) = \langle 0, 0, e_{12}, 0 \rangle \otimes x_0$ and $\text{Ad}_U(\langle 0, 0, e_{21}, 0 \rangle) = \langle 0, 0, e_{21}, 0 \rangle \otimes x_0^*$.

From the second equation of (6.8), we deduce that $\sum_j (T_m)_{i,j} (\beta_{11}^{2,k,j})_{sm}^* = \delta_{m,s} \delta_{i,k} x_0$.

Thus, $\sum_j (T_m)_{i,j} (X(s, m))_{k,j}^* = \delta_{m,s} \delta_{i,k} x_0$. and in particular $T_m X(s, m)^* = \delta_{m,s} \text{diag}(x_0, x_0, x_0)$, which implies

$$X(s, m) = 0 \text{ if } s \neq m \text{ and } X(s, s) = \text{diag}(x_0^*, x_0^*, x_0^*) T_s,$$

which translates into

$$\beta_{11}^{2,i,j} = \text{diag}(x_0^*(T_1)_{i,j}, \dots, x_0^*(T_n)_{i,j}). \quad (6.9)$$

Moreover, as $C^* \beta_{jk}^2 C$ is diagonal from 4. of Lemma 6.5, we get for all $j, k = 1, 2, 3$,

$$\sum_m C_{rm} \overline{C}_{sm}(T_m)_{j,k} = 0 \text{ if } r \neq s.$$

Now we compute

$$\begin{aligned} \text{Ad}_U(\langle 0, 0, 0, e_{rs} \rangle) &= \sum_{j,a,c} e_{11} \otimes e_{j+1,c+1} \otimes (e_{22} + e_{33}) \otimes e_{aa} \otimes ((T_a)_{j,r})^* (T_a)_{c,s} \\ &+ \sum_{j,a,c,p,b} e_{22} \otimes e_{j+1,c+1} \otimes (e_{22} + e_{33}) \otimes e_{ap} \otimes (\beta_{22}^{2,j,r})_{a,b} (\beta_{22}^{2,c,s})_{p,b}^*. \end{aligned}$$

Coinvariance of $M_3(\mathbb{C})$ gives the following relations for all j, r, c, s :

$$\sum_b (\beta_{22}^{2,j,r})_{a,b} (\beta_{22}^{2,c,s})_{p,b}^* \text{ is 0 unless } a = p, \quad (6.10a)$$

$$\sum_b (\bar{\beta}_{11}^{2,j,r})_{a,b} (\beta_{11}^{2,c,s})_{p,b} \text{ is 0 unless } a = p, \quad (6.10b)$$

$$(T_a)_{j,r}^* (T_a)_{c,s} = (T_b)_{j,r}^* (T_b)_{c,s} \quad \forall a, b, \quad (6.10c)$$

$$(T_a)_{j,r}^* (T_a)_{c,s} = \sum_b (\beta_{22}^{2,j,r})_{a,b} (\beta_{22}^{2,c,s})_{a,b}^* \quad \forall a, \quad (6.10d)$$

$$\sum_b (\beta_{22}^{2,j,r})_{a,b} (\beta_{22}^{2,c,s})_{a,b}^* = \sum_b (\bar{\beta}_{11}^{2,j,r})_{a',b} (\beta_{11}^{2,c,s})_{a',b} \quad \forall a, a'. \quad (6.10e)$$

However, it turns out that (6.10c) is the only new information. Indeed, (6.10a) and (6.10b) are consequences of the facts that $\beta_{11}^{2,i,j}$ is diagonal ((6.9)) and $\beta_{22}^{2,j,r} = \bar{\beta}_{11}^{2,j,r}$ (part 2. of Lemma 6.5). The equation (6.10d) follows again from (6.9). Finally (6.10e) follows from (6.10c) and (6.10d) taken together. The equations (6.10a) - (6.10e) show that $\text{Ad}_U(\langle 0, 0, 0, e_{rs} \rangle)$ is given by (6.7c). This completes the proof. \square

We are now in the position to prove Proposition 3.4, i.e. that the universal object in category \mathfrak{C}_J is the CQG given in Lemma 3.3. with corepresentation U as in (3.13).

Proof of Proposition 3.4.

The proof is in two steps: 1. we need to prove that the CQG in Lemma 3.3 with corepresentation (3.13) is an object of the category \mathfrak{C}_J and 2. we need to prove that this object is universal.

1. First we notice that the operator U in (3.13) is indeed a unitary corepresentation: the unitaries/biunitaries $x_k, x_0 x_k, T_m, x_0^* T_m, V$ and their “bar” define unitary corepresentations due to (3.11), and they coact on orthogonal subspaces of H_F in (3.13) so that U is an orthogonal direct sum of unitary corepresentations.

Since our U is of the form (6.1), by Lemma 6.1 it satisfies the compatibility conditions with γ_F and J_F .

Since U is of the form (6.2), with parameters

$$\begin{aligned} \alpha_{11}^1 &:= \sum_{k=1}^n e_{kk} \otimes x_0 x_k, & \alpha_{22}^1 &:= \sum_{i,j=1}^n e_{ij} \otimes V_{ij}, & \alpha_{11}^2 &:= (\alpha_{22}^2)^* := \sum_{k=1}^n e_{kk} \otimes x_k, \\ \beta_{22}^{1,j,k} &:= \sum_{m=1}^n e_{mm} \otimes (T_m)_{j,k}, & \beta_{11}^{1,j,k} &:= (\beta_{22}^{1,j,k})^*, & \beta_{11}^{2,j,k} &:= \sum_{m=1}^n e_{mm} \otimes x_0^* (T_m)_{j,k}, \\ \alpha_{j_1, k_1}^i &:= \beta_{j_1, k_1}^{i, j_0, k_0} := 0, & & \text{if } j_1 \neq k_1, \end{aligned}$$

satisfying 1. – 4. of Lemma 6.5, by Lemmas 6.2 and 6.5 it follows that U commutes with D_F .

Since the parameters defined above satisfy the conditions in Lemma 6.7 too, we have that the adjoint coaction preserves B_F and then our CQG with corepresentation (3.13) is an object of the category \mathfrak{C}_J .

2. Now we pass to universality. From Lemmas 6.1-6.7 it follows that any object (Q, U) in the category \mathfrak{C}_J must be generated by the matrix entries of a corepresentation U of the form (3.13), with matrix entries satisfying (3.10). In particular (3.10b) coincides with (6.6c), (3.10c) coincides

with (6.6d), and (3.10a) coincides with point 3. of Lemma 6.5 after the parameter substitution in (6.6a–6.6b) and after renaming V the matrix α_{22}^1 . Different summands in (3.13) coact on orthogonal subspaces of H_F : hence from unitarity of U we deduce the unitarity of the x_k 's and of the matrices T_m, V and their “bar”, i.e. they must be biunitary. This proves that any object in the category is a quotient of the CQG in Prop. 3.4. \square

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References

- [1] Q.R. Ahmad et al. (SNO Collaboration), *Direct Evidence for Neutrino Flavor Transformation from Neutral-Current Interactions in the Sudbury Neutrino Observatory*, Phys. Rev. Lett. **89** (2002), 011301.
- [2] T. Banica, *Le groupe quantique compact libre $U(n)$* , Commun. Math. Phys. **190** (1997), 143–172.
- [3] T. Banica, *Quantum automorphism groups of small metric spaces*, Pacific J. Math. **219** (2005), 27–51.
- [4] T. Banica, *Quantum automorphism groups of homogeneous graphs*, J. Funct. Anal. **224** (2005), 243–280.
- [5] T. Banica and R. Vergnioux, *Invariants of the half-liberated orthogonal group*, to appear in Ann. Inst. Fourier, arxiv:0902.2719v2 [math.QA].
- [6] J. Bhowmick, D. Goswami and A. Skalski, *Quantum Isometry Groups of 0-Dimensional Manifolds*, to appear in Trans. AMS, arXiv:0807.4288v1 [math.OA].
- [7] J. Bhowmick and D. Goswami, *Quantum Group of Orientation preserving Riemannian Isometries*, J. Funct. Anal. **257** (2009), 2530–2572.
- [8] J. Bhowmick and D. Goswami, *Quantum isometry groups of the Podles spheres*, J. Funct. Anal. **258** (2010), 2937–2960.
- [9] J. Bhowmick and D. Goswami, *Some counterexamples in the theory of quantum isometry groups*, to appear in Lett. Math. Phys., arXiv:0910.4713v1 [math.OA].
- [10] J. Bhowmick and A. Skalski, *Quantum isometry groups of noncommutative manifolds associated to group C^* -algebras*, to appear in J. Geom. Phys., arXiv:1002.2551v2 [math.OA].
- [11] J. Bichon, *Quantum automorphism groups of finite graphs*, Proc. Amer. Math. Soc. **131** (2003), no. 3, 665–673.
- [12] A.H. Chamseddine and A. Connes, *The Spectral Action Principle*, Commun. Math. Phys. **186** (1997), 731–750.
- [13] A.H. Chamseddine and A. Connes, *Why the Standard Model*, J. Geom. Phys. **58** (2008), 38–47.
- [14] A.H. Chamseddine, A. Connes and M. Marcolli, *Gravity and the standard model with neutrino mixing*, Adv. Theor. Math. Phys. **11** (2007), 991–1090.
- [15] A. Connes, *Noncommutative Geometry*, Academic Press, 1994.
- [16] A. Connes, *Noncommutative geometry and reality*, J. Math. Phys. **36** (1995), 6194–6231.
- [17] A. Connes, *Noncommutative differential geometry and the structure of space-time*, Proceedings of the Symposium on Geometry, Huggett, S.A. (ed.) et al., pp. 49–80, Oxford Univ. Press, Oxford UK 1998.
- [18] A. Connes, *Noncommutative geometry and the Standard Model with neutrino mixing*, JHEP **11** (2006), 081.
- [19] A. Connes, *On the spectral characterization of manifolds*, 2008, arXiv:0810.2088v1 [math.OA].
- [20] A. Connes and M. Marcolli, *Noncommutative geometry, quantum fields and motives*, Colloquium Publications, vol. 55, AMS, 2008.

- [21] R. Coquereaux, *On the finite dimensional quantum group $M_3 \oplus (M_{2|1}(\Lambda^2))_0$* , Lett. Math. Phys. **42** (1997), 309–328.
- [22] F. D'Andrea, L. Dąbrowski, G. Landi and E. Wagner, *Dirac operators on all Podleś spheres*, J. Noncomm. Geom. **1** (2007), 213–239.
- [23] F. D'Andrea, L. Dąbrowski and G. Landi, *The Isospectral Dirac Operator on the 4-dimensional Orthogonal Quantum Sphere*, Commun. Math. Phys. **279** (2008), 77–116.
- [24] L. Dąbrowski, G. Landi, M. Paschke and A. Sitarz, *The spectral geometry of the equatorial Podleś sphere*, Comptes Rendus Acad. Sci. Paris **340** (2005), 819–822.
- [25] L. Dąbrowski, G. Landi, A. Sitarz, W. van Suijlekom and J.C. Várilly, *The Dirac operator on $SU_q(2)$* , Comm. Math. Phys. **259** (2005), 729–759.
- [26] L. Dąbrowski, F. Nesti and P. Siniscalco, *A Finite Quantum Symmetry of $M(3, \mathbb{C})$* , Int. J. Mod. Phys. **A13** (1998), 4147–4162.
- [27] Y. Fukuda et al. (Super-Kamiokande Collaboration), *Evidence for Oscillation of Atmospheric Neutrinos*, Phys. Rev. Lett. **81** (1998), 1562–1567.
- [28] D. Goswami, *Quantum Group of Isometries in Classical and Noncommutative Geometry*, Commun. Math. Phys. **285** (2009), 141–160.
- [29] D. Goswami, *Quantum Isometry Group for Spectral Triples with Real Structure*, SIGMA **6** (2010), 007.
- [30] D. Kastler, *Regular and adjoint representation of $SL_q(2)$ at third root of unit*, CPT internal report (1995).
- [31] F. Lizzi, G. Mangano, G. Miele and G. Sparano, *Fermion Hilbert space and fermion doubling in the non-commutative geometry approach to gauge theories*, Phys. Rev. **D55** (1997), 6357–6366.
- [32] A. Maes and A. Van Daele, *Notes on compact quantum groups*, Nieuw Arch. Wisk. **16** (1998), 73–112.
- [33] P. Podleś, *Symmetries of quantum spaces. Subgroups and quotient spaces of quantum $SU(2)$ and $SO(3)$ groups*, Comm. Math. Phys. **170** (1995), 1–20.
- [34] P.M. Soltan, *Quantum $SO(3)$ groups and quantum group actions on M_2* , J. Noncommut. Geom. **4** (2010), 1–28.
- [35] A. Van Daele and S. Wang, *Universal quantum groups*, International J. Math. **7** (1996), 255–264.
- [36] S. Wang, *Free products of compact quantum groups*, Comm. Math. Phys. **167** (1995), no. 3, 671–692.
- [37] S. Wang, *Quantum Symmetry Groups of Finite Spaces*, Commun. Math. Phys. **195** (1998), 195–211.
- [38] S. Wang, *Structure and Isomorphism Classification of Compact Quantum Groups $A_u(Q)$ and $B_u(Q)$* , J. Operator Theory **48** (2002), 573–583.
- [39] S. Wang, *Ergodic actions of universal quantum groups on operator algebra*, Comm. Math. Phys. **203** (1999), no. 2, 481–498.
- [40] S.L. Woronowicz, *Compact matrix pseudogroups*, Commun. Math. Phys. **111** (1987), 613–665.
- [41] S.L. Woronowicz, *Compact quantum groups*, in Symétries quantiques (Les Houches, 1995), pp. 845–884, edited by A. Connes et al., Elsevier, Amsterdam, 1998.

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